§11.1—Parametric Equations

The equations $x = x(t)$ and $y = y(t)$ trace out a curve in the xy-plane as t varies. We often think of the parameter t as time so that the equations represent the path of a particle moving along the curve, and we frequently write the trajectory in the form

$$c(t) = (x(t), y(t)).$$

Any curve defined by a function $y = f(x)$ can be expressed using the parametric equations $x = t$ and $y = f(t)$, and we can sometimes reverse this process by eliminating the parameter t to recover an equation in terms of x and y.

Example 1. Describe the curve traced out by the parametric equations $x = 2t$ and $y = 1 - 6t$.

Example 2. Find parametric equations for the line of slope 2 passing through the point $(3, 5)$.

Parametrizing a line. By generalizing the reasoning from Examples 1 and 2, we find that the equations $x = a + bt$, $y = c + dt$ parametrize a line of slope d/b passing through the point (a, c). Conversely, a line of slope m passing through (x_0, y_0) can be parametrized by $x = x_0 + t$, $y = y_0 + mt$.

Example 3. Describe the curve traced out by the parametric equations

$$x = 3\cos t, \qquad y = 3\sin t \qquad (0 \leqslant t \leqslant 2\pi).$$

Example 4. Find parametric equations for the top half of the circle of radius 7 centered at the point $(5, 4)$.

Slopes of parametric curves. Locally, we can think of the equations $x = x(t)$ and $y = y(t)$ as defining y implicitly as a function of x, which is in turn a function of t. Hence the Chain Rule shows that $dy/dt = (dy/dx)(dx/dt)$ and thus

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)},$$

provided that $x'(t) \neq 0$.

Example 5. Find the slope of the tangent line to the curve $c(t) = (t^3, \sin 2t)$ at the point $t = \pi$.

§11.2—Arc Length and Speed

Suppose we want to compute the length of a curve in the plane. If we think of ds as the length of a small piece of the curve, then we can use the Pythagorean Theorem to write

$$ds \approx \sqrt{(dx)^2 + (dy)^2}.$$

If the curve is defined by parametric equations $c(t) = (x(t), y(t))$, and is traversed exactly once as t increases from a to b, then we have $dx = x'(t)dt$ and $dy = y'(t)dt$, so the length is

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Example 1. Use the arc length formula to find the circumference of a circle of radius R.

Speed. The speed of a particle moving along a curve is given by the derivative of the distance traveled (which is the arc length). Hence by the Fundamental Theorem of Calculus, we find that the speed of the particle with trajectory $c(t) = (x(t), y(t))$ is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}.$$

Note that this is just the length of the velocity vector $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$.

Example 2. Find the speed of a particle with trajectory $c(t) = (e^{3t}, t^2 + 4t + 1)$ when $t = 0$.

Surface Area. We can find the surface area of revolution for a curve with parametric equations by using a formula similar to the arc length integral. The adjustment is that we multiply the arc length element ds by $2\pi r$, where r is the distance from the curve to the axis of revolution, to get the surface area of a thin band. Hence if the curve $c(t) = (x(t), y(t))$ on the interval $[a, b]$ is revolved about the x-axis, the area of the resulting surface is

$$S = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Example 3. A cycloid representing the trajectory of a point on the circumference of a rolling unit circle has parametric equations

$$c(t) = (t - \sin t, 1 - \cos t).$$

Find the area of the surface generated by revolving one arch of this cycloid about the x-axis.

§11.3—Polar Coordinates

Fix an origin O and an initial ray from O. A point P can be represented by the polar coordinates (r, θ), where r is the distance from O to P and θ is the angle from the initial ray to the ray OP.

If we choose the origin O to be the point $(0, 0)$ in the xy-plane and the initial ray to be the positive x-axis, then we can convert between polar and rectangular coordinates as follows:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

It is sometimes useful to allow negative values of r, and we define $(-r, \theta)$ to be the reflection of (r, θ) through the origin. For example, $(2, 7\pi/6)$ and $(-2, \pi/6)$ represent the same point.

Example 1. Convert the point $(r, \theta) = (4, \pi/3)$ to rectangular coordinates.

Example 2. Find three different polar coordinate representations of the point $(x, y) = (-1, 1)$.

Example 3. Describe the curves represented by the following polar equations.

(a) $r = 3$ (b) $\theta = \dfrac{\pi}{6}$

Example 4. Find a polar equation for the circle $x^2 + (y - 2)^2 = 4$.

Example 5. Find a Cartesian equation for the curve $r = \dfrac{4}{3\cos\theta - \sin\theta}$.

Graphing in Polar Coordinates. Notice that a polar curve $r = f(\theta)$ can be viewed in terms of the parametric equations $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Hence the slope is given by $dy/dx = y'(\theta)/x'(\theta)$, and this observation can be useful in curve-sketching. For our purposes, however, it usually suffices to plot a selection of well-chosen points. Most graphing calculators also have a polar mode that can be used to visualize such curves.

Example 6. Sketch the graph of the polar curve $r = 1 - \cos\theta$.

§12.1—Vectors in the Plane

A vector lying in the xy-plane is determined by an initial point $P = (a_1, b_1)$ and a terminal point $Q = (a_2, b_2)$. The horizontal and vertical components of the vector $\overrightarrow{PQ}$ are $a_2 - a_1$ and $b_2 - b_1$, and we write $\overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1 \rangle$.

We view this as equivalent to a vector whose initial point is the origin and whose terminal point is $(a_2 - a_1, b_2 - b_1)$. Vector addition is performed by adding component-wise, and scalar multiplication is performed by multiplying each component by the given scalar.

Example 1. Given that $\mathbf{u} = \langle 4, 3 \rangle$ and $\mathbf{v} = \langle 2, -1 \rangle$, calculate each of the following.

(a) $\mathbf{u} + \mathbf{v}$

(b) $2\mathbf{u} - 3\mathbf{v}$

We say that $\mathbf{u}$ is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$ if we can write $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$ for some real numbers (scalars) s and t. The vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ are known as the standard basis vectors in $\mathbf{R}^2$, and every vector can be expressed as a linear combination of $\mathbf{i}$ and $\mathbf{j}$ as follows:

$$\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}.$$

Example 2. Express $\mathbf{u} = \langle 5, 7 \rangle$ as a linear combination of $\mathbf{v} = \langle 2, 1 \rangle$ and $\mathbf{w} = \langle 1, -1 \rangle$.

The length (or magnitude) of $\mathbf{v}$ is given by $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$, and a vector with length 1 is called a **unit vector**. We sometimes write $\mathbf{e_v} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ for the unit vector in the direction of $\mathbf{v}$.

Example 3. Find a unit vector in the direction of $\mathbf{v} = \langle 3, -2 \rangle$.

§12.2—Vectors in Three Dimensions

Coordinates and distance in three-space. We label the coordinate axes in $\mathbf{R}^3$ to satisfy the right-hand rule: if you curl the fingers of your right hand from the positive x-axis towards the positive y-axis, your thumb should point along the positive z-axis. The distance between $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is

$$|P - Q| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 1. Describe the points in $\mathbf{R}^3$ defined by the following equations.

(a) $(x - 1)^2 + y^2 + (z + 3)^2 = 25$

(b) $(x + 2)^2 + (y - 1)^2 = 9$

A vector in three dimensions can be expressed as

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

where $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are the standard basis vectors for $\mathbf{R}^3$. The magnitude of $\mathbf{v}$ is now given by $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$, and vector addition and scalar multiplication are again defined componentwise by

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \qquad \text{and} \qquad \lambda\mathbf{v} = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle.$$

Example 2. Calculate $2\langle 3, 4, -1 \rangle + 3\langle 1, 0, 5 \rangle$.

Example 3. Given the points $P = (1, 2, 0)$ and $Q = (-3, 0, 5)$, find a unit vector in the direction of $\mathbf{v} = \overrightarrow{PQ}$.

Equation of a line in space. If a line passes through (x_0, y_0, z_0) and $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is any vector parallel to the line, then

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

is a vector from the origin to an arbitrary point on the line. In other words, the line has parametric equations

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$$

Example 4. Find parametric equations for the line through the points $P = (1, 2, 0)$ and $Q = (3, 1, -4)$.

Example 5. Find the point of intersection (if one exists) of the lines $\mathbf{r}_1(t) = \langle 2, 1, 1 \rangle + t\langle -4, 0, 1 \rangle$ and $\mathbf{r}_2(s) = \langle -4, 1, 5 \rangle + s\langle 2, 1 - 2 \rangle$.

§12.3—The Dot Product

The dot product (or scalar product) of the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined by the formula

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example 1. Compute $\mathbf{u} \cdot \mathbf{v}$, where $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle -5, 4, 2 \rangle$.

Note that the dot product of two vectors is a scalar, not a vector! The dot product is obviously commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, and it is easy to check that it distributes over vector addition: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$. It is also useful to observe that $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

It turns out that the dot product also may be computed using the formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta,$$

where θ is the angle between the two vectors. When vectors are given in component form, we typically use the first formula to compute the dot product and then use the second to find the angle between the vectors. Note that two nonzero vectors are **orthogonal** (or perpendicular) if and only if their dot product is zero.

Example 2. Find the angle between the vectors $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$.

Example 3. Find a unit vector orthogonal to $5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Work. If a force vector $\mathbf{F}$ is applied to move an object along the displacement vector $\mathbf{d}$, then the work done is $\mathbf{F} \cdot \mathbf{d}$, since $\|\mathbf{F}\| \cos\theta$ is the component of the force in the direction of motion.

Example 4. A force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j}$ is exerted to move an object up a ramp along the displacement vector $\mathbf{d} = 5\mathbf{i} + \mathbf{j}$. Find the work done on the object.

Projections. In the above instance, the quantity $\|\mathbf{F}\| \cos\theta$ is called the scalar component of $\mathbf{F}$ in the direction of $\mathbf{d}$. In general, the component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is given by

$$\|\mathbf{u}\| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \mathbf{u} \cdot \mathbf{e}_{\mathbf{v}}.$$

§12.4—The Cross Product

The cross product (or vector product) of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = (\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta)\mathbf{n},$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and where $\mathbf{n}$ is the unit vector perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$ whose direction is given by the right-hand rule. Note that the cross product of two vectors is a vector, whereas the dot product of two vectors is a scalar. Also note that two nonzero vectors are parallel if and only if their cross product is the zero vector.

Torque. If we turn a bolt by applying a force $\mathbf{F}$ to a wrench with lever arm $\mathbf{r}$, we produce a torque vector given by $\mathbf{r} \times \mathbf{F}$. Note that the magnitude of the torque is the length of the lever arm times the component of force in the perpendicular direction, $\|\mathbf{F}\| \sin\theta$.

Properties. Unlike the dot product, the cross product is not commutative. In fact,

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}).$$

However, the cross product (like the dot product) does distribute over addition:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \qquad \text{and} \qquad (\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u}),$$

we can factor out scalars: $(r\mathbf{u}) \times (s\mathbf{v}) = rs(\mathbf{u} \times \mathbf{v})$. These properties, together with observations about cross products of the standard basis vectors ($\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{i} \times \mathbf{i} = \mathbf{0}$, *etc.*) lead to the component formula for cross product given on the next page. A similar method can be used to establish the component formula for dot product.

The component formula. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k},$$

where the 2×2 determinants are given by the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Example 1. Given $\mathbf{u} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, calculate the cross product $\mathbf{u} \times \mathbf{v}$.

A very useful property of the cross product is that it produces a vector perpendicular to the plane of the two given vectors.

Example 2. Find a vector perpendicular to the plane determined by the points $P(1, 1, 1)$, $Q(2, 1, 3)$, and $R(3, -1, 1)$.

Area. The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta.$$

Example 3. Find the area of the triangle with vertices $P(-2, 2, 0)$, $Q(0, 1, -1)$, and $R(-1, 2, -2)$.

Triple scalar product and volume. The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is the absolute value of the expression

$$\|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos\theta = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}),$$

where θ is the angle between $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$.

Example 4. Find the volume of the parallelepiped determined by the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = -\mathbf{i} - \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

§12.5—Planes in Three-Space

Equation of a plane. Suppose that a plane contains the point $P_0(x_0, y_0, z_0)$ and that the vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal (perpendicular) to the plane. Then if $P(x, y, z)$ is an arbitrary point in the plane and $\mathbf{v}$ is the vector from P_0 to P, we must have

$$\mathbf{n} \cdot \mathbf{v} = A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Example 1. Find the equation of the plane through the point $(2, 1, -3)$ normal to the vector $\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

Example 2. Find parametric equations for the line through the point $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$.

Example 3. Find the equation of the plane through $P(2, 4, 5)$, $Q(1, 5, 7)$, and $R(-1, 6, 8)$.

Example 4. Find the angle between the planes $5x + y - z = 10$ and $x - 2y + 3z = -1$.

Example 5. Find the point in which the line $x = -1 + 3t$, $y = -2$, $z = 5t$ intersects the plane $2x - 3z = 7$.

Example 6. Find parametric equations for the line of intersection of the planes $x + y + z = 1$ and $x + 2y + 3z = 1$.

§12.7—Cylindrical and Spherical Coordinates

Cylindrical coordinates. If we express x and y in polar coordinates and leave z unchanged, the coordinates (r, θ, z) are called cylindrical coordinates. The equations for converting between Cartesian and cylindrical coordinates (r, θ, z) are therefore the same as those in §11.3:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

Example 1. Find an equation in cylindrical coordinates of the form $z = f(r, \theta)$ for the hemisphere defined by $x^2 + y^2 + z^2 = 1$ and $z \geqslant 0$.

Example 2. Find an equation for the cone $z = r$ in rectangular coordinates.

Spherical coordinates. For certain problems, it may be more convenient to represent a point P using the spherical coordinates (ρ, θ, ϕ), where ρ is the distance from P to the origin, θ is the same as in cylindrical coordinates, and ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis. Note that $\rho \geqslant 0$ and $0 \leqslant \phi \leqslant \pi$. We can relate to cylindrical coordinates using the equations

$$r = \rho \sin\phi, \qquad z = \rho\cos\phi, \qquad \rho = \sqrt{r^2 + z^2}$$

and hence to Cartesian coordinates via

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi,$$

$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

Example 3. Convert the point $(3, \pi/6, \pi/4)$ from spherical to rectangular coordinates.

Example 4. Convert the point $(\sqrt{3}, 0, 1)$ from rectangular to spherical coordinates.

Example 5. Describe the set of points represented by each spherical coordinate equation below.

(a) $\rho = 2$ (b) $\phi = \pi/3$

§13.1—Vector-Valued Functions

A vector function of a single variable has the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle x(t), y(t), z(t)\rangle.$$

As t runs over a subset of the real numbers, the vectors $\mathbf{r}(t)$ trace out a curve in $\mathbf{R}^3$. We call $\mathbf{r}(t)$ a vector parametrization of this path, and the equations $x = x(t)$, $y = y(t)$, $z = z(t)$ are parametric equations. Notice that the case where $z(t) = 0$ reduces to curves in $\mathbf{R}^2$, as studied in §11.1. In §12.5, we saw that when $x(t)$, $y(t)$, and $z(t)$ are linear functions of t the curve traced out is a line.

Example 1. Find a vector parametrization of the line passing through $(1, 0, 4)$ and $(4, 1, 2)$.

Example 2. Find a vector parametrization of the circle with radius 3 and center $(2, 1, 5)$ lying in a plane parallel to the yz-plane.

Example 3. Sketch the curve traced out by the function $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ $(0 \leqslant t \leqslant 4\pi)$.

Example 4. Find the domain of the function $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + e^{-3t}\mathbf{j} + \sqrt{t+3}\,\mathbf{k}$.

§13.2—Calculus of Vector-Valued Functions

Familiar concepts from calculus like limits, continuity, derivatives, antiderivatives, and definite integrals extend to vector-valued functions by applying the ordinary definitions componentwise. Many of the usual rules extend in a natural way to this setting.

Example 1. Consider the vector-valued function $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + e^{-3t}\mathbf{j} + (\sin 2t)\mathbf{k}$.

(a) Evaluate $\lim\limits_{t\to 0} \mathbf{r}(t)$.

(b) Compute $\mathbf{r}'(t)$.

(c) Evaluate $\displaystyle\int_0^1 \mathbf{r}(t)\,dt$.

By recalling the definition of the derivative,

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

and the geometric interpretation of vector addition, we find that $\mathbf{r}'(t)$ is always tangent to the curve $\mathbf{r}(t)$. The vector $\mathbf{r}'(t)/\|\mathbf{r}'(t)\|$ is called the **unit tangent vector**.

Example 2. Find the unit tangent vector at $t = 0$ for the curve in Example 1.

Example 3. Find parametric equations for the tangent line to the helix defined by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ at the point $(-1, 0, \pi)$.

Example 4. Verify that the product rule

$$\frac{d}{dt}\big(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)\big) = \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t)$$

holds for the functions $\mathbf{r}_1(t) = t^3\mathbf{i} + 2t\mathbf{j} - 3\mathbf{k}$ and $\mathbf{r}_2(t) = t^{-1}\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$.

Given information about a particle's velocity vector and initial position, we can use integration to find its position function.

Example 5. Solve the initial-value problem

$$\frac{d\mathbf{r}}{dt} = (t^3 + 4t)\mathbf{i} + \frac{1}{1+t^2}\mathbf{j} + 2t^2\mathbf{k}, \qquad \mathbf{r}(1) = \mathbf{i} + \mathbf{j}.$$

§13.3—Arc Length and Speed

The formulas we learned for arc length and speed in §11.2 generalize in the obvious way to curves in three dimensions. If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then the speed is given by

$$\|\mathbf{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2},$$

and the arc length from $t = a$ to $t = b$ is given by

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt.$$

Example 1. Find the length of the curve defined by $\mathbf{r}(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3 \rangle \quad (0 \leqslant t \leqslant 4)$.

Example 2. A particle moves in space according to the position vector

$$\mathbf{r}(t) = (2\cos \pi t)\mathbf{i} + (4e^{-t})\mathbf{j} - (t^2 \ln t)\mathbf{k}.$$

(a) Find the particle's speed at $t = 1$

(b) Find a unit vector that gives the direction of motion at $t = 1$.

§14.1—Functions of Two or More Variables

Many quantities of interest depend on more than just one variable. For instance, a location's temperature or elevation depends on both latitude and longitude. Functions of two variables, say $z = f(x, y)$, can be graphed as surfaces in three-space by plotting the output on the z-axis.

Example 1. Let $f(x, y) = \sqrt{9 - x^2 - y^2}$.

(a) Evaluate $f(1, -2)$.

(b) Find the domain and range of f.

(c) Describe the curve defined by $f(x, y) = 2$.

The set of points where a function $f(x, y)$ has a constant value, as considered in (c) above, is called a **level curve**.

Example 2. Sketch the level curve of the function $f(x, y) = x^2 + y$ that passes through the point $(2, 3)$.

When $f(x, y, z)$ is a function of three variables, its graph lies in four-dimensional space, so we can't draw the graph. However, the set of points where f takes a constant value, say $f(x, y, z) = c$, can be graphed in 3-dimensional space. These **level surfaces** provide a way of depicting some properties of the function.

Example 3. Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

By plotting several level curves of a function, we can create a **contour map** similar to those used in weather reports and hiking elevation guides. The following specific examples are useful to keep in mind:

- Lines of constant temperature (isotherms) on a weather map are level curves of the temperature function $T(x, y)$.
- Lines of constant pressure (isobars) on a weather map are level curves of the barometric pressure function $P(x, y)$.
- Lines of constant elevation on a hiking map are level curves of the function $H(x, y)$ that measures height above sea level.

Rates of Change. Understanding how a function $f(x, y)$ changes as we perturb the input (x, y) requires more care than in the one-variable case, because there are infinitely many directions to consider. Contour maps provide a visual illustration of the fact that the rate of change at a particular point depends on the direction one intends to move. Contour lines that are very close together indicate places where the graph is very steep (that is, where the rate of change in the direction perpendicular to the initial contour is large). Contour lines that are far apart indicate a direction in which the function changes more gradually.

Example 4. Consider the function $f(x, y) = x^2 + 2y^2$.

(a) Find the average rate of change of f with respect to x from $(1, 1)$ to $(3, 1)$.

(b) Find the average rate of change of f with respect to y from $(1, 1)$ to $(1, 3)$.

By graphing the level curve $f(x, y) = c$ in three dimensions in the plane $z = c$, we obtain a **horizontal trace** of the function. Similarly, the curves obtained by intersecting with planes $x = a$ or $y = b$ are called **vertical traces**. These traces can be of assistance in sketching the surface or visualizing rates of change along the x or y axis.

Example 5. Describe the vertical and horizontal traces of the function $f(x, y) = x^2 + y$

§14.2—Limits and Continuity in Several Variables

Limits for functions of two variables are defined in much the same way as the one variable case and have many of the same properties (see Theorem 1 on page 788, for example). The main difference is that in one variable we only have to approach the same value from the right and the left, whereas in two variables there are infinitely many directions to check.

Example 1. Evaluate each of the following limits.

(a) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2+xy+1}{3xy^2+4}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

Just as in the single variable case, we say that a function of two variables is continuous at a point if the function is defined there, the limit exists, and the limit equals the function value. Limits of continuous functions can therefore be evaluated by direct substitution, as in Example 1a. Note that the function considered in Example 1b is undefined at $(0,0)$ and hence is not continuous there.

Example 2. Show that each function below has no limit as $(x,y)\to(0,0)$.

(a) $f(x,y) = \dfrac{x^4}{x^4+y^2}$

(b) $f(x,y) = \dfrac{xy}{x^2+3y^2}$

The Two-Path Test. The results of Example 2 illustrate a general principle: if we can find two different paths approaching the point (a,b), along which a function $f(x,y)$ has two different limits, then the limit of f as $(x,y) \to (a,b)$ does not exist. To show that a limit does exist, however, it does not suffice to produce the same result on two different paths. Here we need a more general argument that applies to all possible paths, as the following example illustrates.

Example 3. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5xy^2}{x^2+y^2} = 0.$

§14.3—Partial Derivatives

We compute the partial derivatives of a function $f(x, y)$ by holding one variable constant and applying the usual definition with respect to the remaining variable. For example, the notation $\partial f/\partial x$ or f_x represents the derivative of f with respect to x, while treating y as a constant:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

We may interpret $f_x(a, b)$ as the instantaneous rate of change of f at (a, b) in the direction $\mathbf{i}$. Geometrically, it gives the slope of the tangent line to the vertical trace of the surface $z = f(x, y)$ in the plane $y = b$ at the point where $x = a$. A similar definition and interpretation applies to the partial derivatives with respect to y and for functions of more than two variables. All of the familiar differentiation rules hold for the computation of partial derivatives.

Example 1. Calculate $f_x(3, 2)$ and $f_y(3, 2)$ for the function

$$f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2.$$

Example 2. The ideal gas law states that $PV = nRT$, where P, V, and T are pressure, volume, and temperature, and where n and R are constants. Compute $\partial P/\partial T$ and $\partial P/\partial V$.

Example 3. Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function $f(x, y) = e^{-y} \sin(3x + 2y)$.

Example 4. Calculate the partial derivatives f_x, f_y, and f_z for the function $f(x, y, z) = xy^{\cos z}$. In which of the directions $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$ is f changing most rapidly at the point $(3, 2, 0)$?

Higher Derivatives. As with ordinary differentiation, we can apply partial differentiation repeatedly. For instance, the notation $\dfrac{\partial^2 f}{\partial x^2}$ or f_{xx} means the second partial derivative of f with respect to x, whereas $\dfrac{\partial^2 f}{\partial y \partial x}$ or f_{xy} differentiates first with respect to x and then with respect to y.

Example 5. Find all the second-order partial derivatives (f_{xx}, f_{xy}, f_{yx}, and f_{yy}) of the function

$$f(x, y) = y + x^2 y + 4y^3 - \ln(y^2 + 1).$$

The Mixed Derivative Theorem. The fact that $f_{xy} = f_{yx}$ in Example 5 is not a coincidence. This is true whenever f_{xy} and f_{yx} are continuous, and a similar result holds for higher-order derivatives. We can sometimes take advantage of this fact to compute mixed partials very efficiently.

Example 6. Compute f_{xyz} for the function $f(x, y, z) = \dfrac{x^3 \sin(e^{xz})}{\sqrt{z^2 + 1}} + x^2 y^3 z^5 + y \arctan z$.

§14.4—Differentiability and Tangent Planes

Recall from MAT 161 that the linearization of a function $f(x)$ at $x = a$ is given by

$$L(x) = f(a) + f'(a)(x - a).$$

The graph of $y = L(x)$ is of course just the tangent line to the curve $y = f(x)$ at the point $x = a$. We generalize this concept by defining the linearization of $f(x, y)$ at (a, b) to be

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Here the graph of $z = L(x, y)$ is the **tangent plane** to the surface $z = f(x, y)$ at the point $(x, y) = (a, b)$. Notice that substituting $x = a$ or $y = b$ reduces this to the tangent line of a vertical trace of the surface. Hence the tangent plane is determined by requiring that it contain the tangent lines to both vertical trace curves.

Example 1. Find the equation for the tangent plane to the surface $f(x, y) = x^2 + y^3$ at the point $(3, 2)$.

Example 2. Find the linearization to the function $f(x, y) = x^5y^2$ at the point $(1, 2)$, and use it to estimate $(1.01)^5(2.02)^2$.

Differentiability. Roughly speaking, we say that f is differentiable at (a, b) if the linearization $f(x, y)$ provides a "good" approximation to $f(x, y)$ near (a, b). A sufficient condition for differentiability is that f_x and f_y exist and are continuous on some open disk. See page 806 for the precise statements. When estimating change for a differentiable function f, it is often convenient to write $\Delta f = f(x, y) - f(a, b)$. Then the approximation $f(x, y) \approx L(x, y)$ allows us to write

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

Example 3. Body Mass Index (BMI) is defined by the equation $I = W/H^2$, where W represents body weight (in kilograms) and H represents height (in meters). Suppose a boy is currently 1.3 meters tall and weighs 40 kg. If he gains 5 kg, approximately how much taller would he have to get in order to keep his BMI the same?

Analogous formulas apply for functions of three or more variables; for instance,

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

is the linearization of a function $f(x, y, z)$ at the point (a, b, c), and we get the corresponding approximate change formula

$$\Delta f \approx f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z.$$

Example 4. Find the linearization of the function $f(x, y, z) = xz^3 \ln y$ at the point $(5, e, 2)$.

§14.5—The Gradient and Directional Derivatives

The **gradient vector** of the function $f(x, y, z)$ is defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Note that for a function $f(x, y)$ of only two variables the $\mathbf{k}$ component vanishes. We frequently write ∇f_P to stand for the particular vector obtained by evaluating the gradient at the point P.

Example 1. Compute the gradient vector of each function at the indicated point.

(a) $f(x, y) = x/y \quad P = (2, 3)$

(b) $f(x, y, z) = x^3 y\sqrt{z} \quad P = (2, 3, 4)$

Directional derivatives. The gradient vector is important because it encodes information about the rates of change of the function $f(x, y, z)$. In particular, the **directional derivative** of f along the unit vector $\mathbf{u}$, written $D_{\mathbf{u}}f$, is the rate of change of f with respect to distance along a line $\mathbf{c}(t)$ parallel to $\mathbf{u}$. Such a line has parametric equations of the shape

$$x = x_0 + tu_1, \quad y = y_0 + tu_2, \quad z = z_0 + tu_3,$$

so a natural generalization of the Chain Rule gives

$$D_{\mathbf{u}}f \;=\; \nabla f \cdot \mathbf{c}'(t) = \nabla f \cdot \mathbf{u} \;=\; \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

Notice that the derivatives in the directions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are just the respective partial derivatives, as we observed in §14.3.

Important: Although the definition of $D_{\mathbf{u}}f$ makes sense for any non-zero vector $\mathbf{u}$, the concept of directional derivative requires that $\mathbf{u}$ be a unit vector.

Example 2. Find the directional derivative of the function $f(x, y) = 2x^2 + y^2$ at the point $(-1, 1)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Example 3. Find the directional derivative of the function $f(x, y, z) = \cos xy + e^{yz} + \ln zx$ at the point $(1, 0, \frac{1}{2})$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Notice that a function increases most rapidly in the direction of the gradient vector and decreases most rapidly in the direction opposite the gradient. Similarly, the function experiences zero change in any direction orthogonal to the gradient. In other words, the gradient at a point is orthogonal to the level curve or surface through that point.

Example 4. Consider the function $f(x, y) = x^2 + y^2$. Determine the directions in which f increases/decreases most rapidly at the point $(1, 2)$ and the directions of zero change at this point.

Recall from last time that the directional derivative of f in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \cos\theta,$$

where $\mathbf{u}$ is a unit vector and θ is the angle between ∇f and $\mathbf{u}$. As a consequence, f increases most rapidly in the direction of ∇f, and the value of the derivative in this direction is $\|\nabla f\|$. Similarly, f decreases most rapidly in the direction $-\nabla f$, and has zero derivative in any direction orthogonal to ∇f.

Example 5. Consider the function $f(x, y, z) = x/y - yz$.

(a) Find the direction in which f increases most rapidly at the point $(4, 1, 1)$, and find the derivative in that direction.

(b) In what directions from the point $(4, 1, 1)$ does f experience zero change?

Tangent planes to level surfaces. The reasoning of Example 5b may be generalized as follows. Since a direction of zero change for the function $f(x, y, z)$ must be orthogonal to the gradient vector ∇f, every curve on the level surface $f(x, y, z) = k$ passing through the point P has its tangent (or velocity) vector orthogonal to ∇f_P. We therefore call the plane through P with normal vector ∇f_P the **tangent plane** to the surface $f(x, y, z) = k$ at the point $P = (a, b, c)$. Since

$$\nabla f_P = f_x(a, b, c)\mathbf{i} + f_y(a, b, c)\mathbf{j} + f_z(a, b, c)\mathbf{k},$$

we know from §12.5 that the equation of the tangent plane is

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.$$

The special case of a surface $z = g(x, y)$ is handled by viewing it as the level surface $f(x, y, z) = g(x, y) - z = 0$, so that the same equation applies with $f_x = g_x$, $f_y = g_y$, and $f_z = -1$, and we recover the results presented in §14.4.

Example 6. Find the equation of the tangent plane to the surface $x^2 + y - z^3 = 21$ at the point $(5, 4, 2)$.

The Chain Rule for Paths. If we have parametric equations for x, y, and z as functions of t, it is natural to compute the rate of change of f with respect to t along the path $\mathbf{c}(t) = (x(t), y(t), z(t))$. For this, we have the following version of the chain rule:

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Notice that in the special case where $\mathbf{c}(t) = (x_0 + u_1 t, y_0 + u_2 t, z_0 + u_3 t)$ is a line, we have $\mathbf{c}'(t) = (u_1, u_2, u_3)$, and hence the formula reduces to the one for directional derivatives:

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f_{(x_0, y_0, z_0)} \cdot \mathbf{u}.$$

Example 7. A bug walks along the elliptical helix $\mathbf{c}(t) = (2\cos t, 3\sin t, 4t)$ carrying a tiny thermometer. If the temperature at (x, y, z) is given by $T(x, y, z) = xy^2 + z^3$, how fast is the temperature changing at $t = \pi/2$?

§14.6—The Chain Rule

Suppose that $f(x_1, \dots, x_n)$ is a differentiable function of n variables, which are each in turn differentiable functions of m variables $t_1, \dots, t_m$. Then for $k = 1, \dots, m$, the derivative of f with respect to $t = t_k$ may be computed using the formula

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t}.$$

This follows from a natural extension of the Chain Rule for Paths discussed in §14.5 by holding all independent variables but t fixed to define a path $\mathbf{c}(t) = (x_1(t), \dots, x_m(t))$. This formula can once again be viewed as a dot product:

$$\frac{\partial f}{\partial t} = \nabla f_{\mathbf{x(t)}} \cdot \frac{\partial \mathbf{x}}{\partial t}.$$

Note that if all but one of the variables x_i are independent of t then this reduces to the ordinary chain rule from single-variable calculus: $\frac{d}{dt}[f(x(t))] = f'(x(t))x'(t)$.

Example 1. Draw a tree diagram and write down the Chain Rule for computing $\partial w/\partial r$ if $w = f(x, y, z, v)$, $x = g(r, s)$, $y = h(r, s)$, $z = j(r, s)$, and $v = k(r, s)$.

Example 2. Consider the function $f(x, y) = y \sin x$, where $x = u^2 + v^2$ and $y = uv^3$.

(a) Use the Chain Rule to compute $\dfrac{\partial f}{\partial v}$.

(b) Verify the result of part (a) by first expressing f as a function of u and v and then differentiating.

Example 3. Consider the function $w = \sqrt{y} + x \tan^{-1} z$, where $x = e^r + t \ln s$, $y = 3rs + t$, and $z = 5s - 2t$. Evaluate $\partial w / \partial s$ at the point $(r, s, t) = (0, 1, 2)$.

Example 4. Consider the function $w = xy^2 + ye^{3z}$, where $x = r + 4s$, $y = 2r - s$, and $z = rs$. Evaluate $\partial w / \partial r$ and $\partial w / \partial s$ at the point $(r, s) = (1, 0)$.

Implicit Differentiation. We may view the level surface $F(x, y, z) = 0$ as implicitly defining a function $z = f(x, y)$, at least locally near a particular point. The chain rule then shows that we can find $\partial z / \partial x$ or $\partial z / \partial y$ by differentiating both sides with respect to the variable in question and then solving algebraically for the desired partial derivative.

Example 5. Find the value of $\dfrac{\partial z}{\partial x}$ at the point $(1, 0, 3)$ if the equation

$$xz + y \ln z - z^2 + 6 = 0$$

defines z implicitly as a function of x and y.

§14.7—Optimization in Several Variables

A **critical point** (a, b) of a function $f(x, y)$ is a point where $f_x(a, b) = f_y(a, b) = 0$ or where at least one of $f_x(a, b)$ and $f_y(a, b)$ fails to exist. Critical points are the only *possible* places where local extrema can occur.

Second Derivative Test. Suppose that $f_x(a, b) = f_y(a, b) = 0$ and that f_{xx}, f_{yy}, and f_{xy} are continuous near (a, b), and let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

(i) If $f_{xx}(a, b) > 0$ and $D(a, b) > 0$ then f has a local minimum at (a, b).

(ii) If $f_{xx}(a, b) < 0$ and $D(a, b) > 0$ then f has a local maximum at (a, b).

(iii) If $D(a, b) < 0$ then f has a saddle point at (a, b).

(iv) If $D(a, b) = 0$ then the test is inconclusive.

The proof of this test is difficult, but the examples $f(x, y) = x^2 + y^2$ and $f(x, y) = x^2 - y^2$, in which $f_{xy} = 0$, provide some useful insight for the simplest cases.

Example 1. Find the location of all local maxima, local minima, and saddle points.

(a) $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

(b) $f(x, y) = x^2 - 4xy + y^2 + 6y + 2.$

Example 2. Find the location of all local maxima, local minima, and saddle points of the function $f(x,y) = x^3 - 3xy + y^3$.

Global extrema. A continuous function on a closed, bounded domain always attains an absolute maximum and an absolute minimum value. These can only occur at critical points or boundary points, so we just find all candidates and compare the function values.

Example 3. Find the global maximum and minimum values of the function $f(x,y) = x^3 - 3xy + y^3$ on the domain $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 3$, and state where these values occur.

§14.8—Lagrange Multipliers: Optimizing with a Constraint

Suppose that the function $f(x, y)$ has a local maximum or minimum value at a point P on the curve $g(x, y) = 0$. If this constraint curve is parametrized by $\mathbf{c}(t) = \langle x(t), y(t)\rangle$, then the Chain Rule gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \mathbf{c}'(t).$$

At a critical point, the above expression is zero, so ∇f must be perpendicular to the curve's tangent vector at P. But we know from §14.5 that ∇g is also perpendicular to this level curve, so we deduce that ∇f and ∇g are parallel. Thus for some real number λ we have

$$\nabla f = \lambda \nabla g.$$

Similar arguments apply to functions of three variables, and in that case one can in fact enforce more than one constraint (see Example 4 in the text).

Example 1. Find the extreme values of $f(x, y) = x^2 y$ lying on the ellipse $x^2 + 2y^2 = 6$.

Example 2. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where the function $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

Example 3. If x, y, and z are positive real numbers satisfying $x + y + z^2 = 16$, what is the largest product the numbers can have?

§15.1—Integration in Several Variables

Suppose we want to calculate the volume of a solid whose base is a given region in the xy-plane and whose upper boundary is some surface $z = f(x, y)$. One approximation strategy is to divide the base into many small rectangles, estimate the height using a sample point in each rectangle, and add up the volumes of the resulting boxes.

Example 1. Estimate the volume of the solid whose base is the rectangle

$$\mathcal{R} = [0, 1] \times [0, 2] = \{(x, y) : \ 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant 2\}$$

and whose upper boundary is the paraboloid $z = 9 - x^2 - y^2$. Divide $\mathcal{R}$ into 8 squares of side length 1/2, and take the sample points to be

(a) the lower left vertices of the squares

(b) the midpoints of the squares

In general, if $\mathcal{R} = [a, b] \times [c, d]$, then we can divide $\mathcal{R}$ into NM rectangles of area $\Delta A = \Delta x \Delta y$, where $\Delta x = (b-a)/N$ and $\Delta y = (d-c)/M$. If $P_{ij} = (x_i, y_j)$ are the sample points, then the volume of the solid with base $\mathcal{R}$ and upper boundary $z = f(x, y)$ is approximated by a double Riemann sum:

$$V \approx S_{N,M} = \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}) \Delta A = \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \Delta x \Delta y.$$

If f is continuous, we can now get the exact volume by taking the limit as $M, N \to \infty$, and this gives rise to a **double integral**:

$$V = \iint_{\mathcal{R}} f(x, y)\, dA.$$

In cases where $f(x, y)$ takes on both positive and negative values, this integral may be interpreted as "signed" volume. Double integrals can commonly be evaluated by converting them to **iterated integrals**, which reflect the decomposition of the solid into cross sections parallel to the x or y-axis. This process is illustrated in the next example.

Example 2. Find the exact volume of the solid from Example 1 by evaluating the double integral $\displaystyle\iint_{\mathcal{R}} (9 - x^2 - y^2)\, dA$, where $\mathcal{R} = [0,1] \times [0,2]$.

Fubini's Theorem. The results of Example 2 may be generalized as follows: If $f(x, y)$ is continuous, then $\iint_{\mathcal{R}} f(x, y)\, dA$ can be evaluated by first integrating with respect to y along cross-sections of $\mathcal{R}$ parallel to the y-axis and then integrating with respect to x. Alternatively, one can first integrate with respect to x along cross-sections of $\mathcal{R}$ parallel to the x-axis and then integrate with respect to y. That is, if $\mathcal{R} = [a, b] \times [c, d]$ and f is continuous on $\mathcal{R}$, then

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Example 3. Evaluate $\displaystyle\iint_{\mathcal{R}} \cos(2x + y)\, dA$, where $\mathcal{R} = [0, \pi/4] \times [0, \pi/3]$.

We can sometimes apply Fubini's Theorem to our advantage when one order leads to an easier integration.

Example 4. Evaluate $\displaystyle\int_0^2 \int_0^3 y^3 e^{xy^2}\, dy\, dx$ by reversing the order of integration.

Linearity. As with integrals in a single variable, double integrals satisfy

$$\iint_{\mathcal{R}} (f(x,y) + g(x,y))\, dA = \iint_{\mathcal{R}} f(x,y)\, dA + \iint_{\mathcal{R}} g(x,y)\, dA$$

and

$$\iint_{\mathcal{R}} cf(x,y)\, dA = c \iint_{\mathcal{R}} f(x,y)\, dA$$

for any constant c and any rectangle $\mathcal{R}$. In particular, it follows from the second property that

$$\int_a^b \int_c^d g(x)h(y)\, dy\, dx = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right).$$

Example 5. Evaluate $\displaystyle\int_1^e \int_0^2 \frac{x^3}{y}\, dx\, dy.$

§15.2—Double Integrals over More General Regions

By modifying the Riemann sums introduced in §15.1, we can define double integrals of continuous functions $f(x,y)$ over more general domains $\mathcal{D}$ with piecewise-smooth boundaries. The strategy for evaluation is again to set up an iterated integral that covers the region $\mathcal{D}$ by either horizontal or vertical strips, depending on whether we want to integrate first with respect to x or y.

Example 1. Integrate the function $f(x,y) = xy + 3$ over the region in the first quadrant bounded by the parabola $y = x^2$, the line $x = 0$, and the line $y = 4$.

Example 2. Evaluate $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$ by reversing the order of integration.

Applications of double integrals. Some easy applications of double integrals are:

- $\iint_{\mathcal{D}} 1\, dA$ is the area of $\mathcal{D}$.
- $\iint_{\mathcal{D}} f(x,y)\, dA$ is the signed volume of the solid with base $\mathcal{D}$ and height $z = f(x,y)$.
- $\bar{f} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x,y)\, dA$ is the average value of $f(x,y)$ on $\mathcal{D}$.

Example 3. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 3$.

Example 4. Find the average value of the function $f(x,y) = x^2 e^{xy}$ over the region

$$\mathcal{D} = \{(x,y) :\ y \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}.$$

§15.3—Triple Integrals

The same strategy we applied to double integrals can be used to integrate continuous functions $f(x, y, z)$ over suitably nice regions $\mathcal{W}$ in $\mathbf{R}^3$. Here we integrate with respect to the volume element $dV = dx\, dy\, dz$, and there are 6 possible orders for setting up the iterated integrals. As with double integrals, the simplest case is when $\mathcal{W}$ is a box $\mathcal{B} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.

Example 1. Evaluate $\displaystyle\iiint_{\mathcal{B}} \left(\frac{x \cos z}{y} + y \right) dV$, where $\mathcal{B} = [0, 1] \times [1, e] \times [0, \pi/2]$.

Example 2. Integrate the function $f(x, y, z) = xyz$ over the region $\mathcal{W}$ defined by the inequalities

$$0 \leqslant z \leqslant 1, \quad 0 \leqslant y \leqslant \sqrt{1 - x^2}, \quad 0 \leqslant x \leqslant 1.$$

Example 3. Evaluate $\iiint_{\mathcal{W}} e^z \, dV$, where $\mathcal{W}$ is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

Applications of triple integrals. Two immediate applications of triple integrals are:

- $\iiint_{\mathcal{W}} 1 \, dV$ is the volume of $\mathcal{W}$.
- $\bar{f} = \frac{1}{\text{Vol}(\mathcal{W})} \iiint_{\mathcal{W}} f(x,y,z) \, dV$ is the average value of $f(x,y,z)$ on $\mathcal{W}$.

In the situation of Example 3, for instance, we would integrate the function $f(x,y,z) = 1$ instead of e^z to obtain the volume of the tetrahedron. Dividing the result of Example 3 by the volume of the tetrahedron would then give the average value of the function $f(x,y,z) = e^z$ over this region. We will investigate further applications of triple integrals to mass, center of mass, moments of inertia, and probability theory in §15.5.

Example 4. Let $\mathcal{W}$ be the region in the first octant that is common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

(a) Find the volume of $\mathcal{W}$.

(b) Find the average value of the function $f(x, y, z) = yz$ on $\mathcal{W}$.

§15.4—Integration in Polar, Cylindrical and Spherical Coordinates

Polar and cylindrical coordinates. For functions and/or regions with radial symmetry, we can often simplify a double integral by converting to polar coordinates using the basic equations

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

However, the area element $dA = dx\,dy$ must be replaced by

$$dA = r\,dr\,d\theta$$

because the arc length corresponding to the angle $d\theta$ at a distance r from the origin is given by $r\,d\theta$. Similarly, the volume element $dV = dx\,dy\,dz$ is replaced by

$$dV = r\,dz\,dr\,d\theta$$

for triple integrals in cylindrical coordinates.

Example 1. Evaluate the integral $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy$.

Example 2. Find the average height of the cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \leqslant a^2$ in the xy-plane.

Example 3. Evaluate the integral $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Example 4. Integrate the function $f(x, y, z) = \frac{1}{4}z$ over the solid region bounded by the paraboloids $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$.

Spherical coordinates. For certain problems involving spheres and cones, it may be more convenient use spherical coordinates (ρ, θ, ϕ). Recall from §12.7 that ρ is the distance from the point P to the origin, ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis, and θ is the same as in cylindrical coordinates, so that

$$r = \rho \sin \phi, \qquad z = \rho \cos \phi, \qquad \rho = \sqrt{r^2 + z^2}$$

and

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad \rho = \sqrt{x^2 + y^2 + z^2}.$$

The volume element in spherical coordinates is

$$dV = d\rho \times r\, d\theta \times \rho\, d\phi = \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta.$$

This formula is obtained by considering a spherical wedge that can be approximated as a cube with side lengths $d\rho$, $\rho\, d\phi$, and $r\, d\theta = \rho \sin \phi\, d\theta$.

Example 5. Use spherical coordinates to compute the volume of a sphere of radius R.

Example 6. Find the volume of the "ice cream cone" bounded above by the sphere $x^2+y^2+z^2 = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$.

§15.5—Applications of Multiple Integrals

Density functions. Integrating a density function (such as mass or poplulation density) over a given region yields the total amount (*e.g.* the total mass or total population) present in the region. For instance, the result of Example 4 in §15.4 may be viewed as the mass of the solid lying between the two paraboloids and having density $\delta(x, y, z) = \frac{1}{4}z$.

Example 1. A city has population density $\delta(x, y) = 2000(x^2 + y^2)^{-0.2}$ people per square kilometer, where the city center is located at the origin $(0, 0)$. Find the total population within a 4-kilometer radius of the center.

Example 2. Find the mass of a cube that is bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$, and has density given by $\delta(x, y, z) = x + 2y + 3z$ kilograms per cubic unit.

Example 3. Find the mass of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the plane $z = 2$, if the solid has constant density $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ kilograms per cubic unit.

Moments and centers of mass. Suppose that a region $\mathcal{W} \subseteq \mathbf{R}^3$ is occupied by an object with density $\delta(x, y, z)$ and total mass M. The the x-coordinate of the center of mass is given by

$$\bar{x} = \frac{1}{M} \iiint_{\mathcal{W}} x\delta(x, y, z)\, dV,$$

with analogous formulas for $\bar{y}$ and $\bar{z}$. If the density is constant, then the center of mass depends only on the geometry of $\mathcal{W}$ and is called the **centroid** of the region. Identical formulas involving double integrals hold for two-dimensional objects. The integral appearing in the formula for $\bar{x}$ above is an example of a *moment*, and similar formulas (with quadratic instead of linear weights) are used to compute moments of inertia.

Example 4. Find the center of mass of a cylinder of radius 2, height 4, and mass density $\delta(x, y, z) = \sqrt{z}$, where z is the height above the base.

Probability. A density function $p(x, y)$ with the property that $\iint_{\mathbf{R}^2} p(x, y)\, dA = 1$ is called a joint probability density function. Such functions allow us to measure the likelihood that two continuous random variables, X and Y take on certain values. For example, the probability that $a \leqslant X \leqslant b$ and $c \leqslant Y \leqslant d$ is given by

$$P(a \leqslant X \leqslant b; c \leqslant Y \leqslant d) = \int_a^b \int_c^d p(x, y)\, dy\, dx.$$

More generally, the integral of $p(x, y)$ over a region $\mathcal{D} \in \mathbf{R}^2$ gives the probability that $(X, Y) \in \mathcal{D}$.

Example 5. Suppose that X and Y represent the number of months it takes for two different sensors in an aircraft to fail, and that their joint probability density function is given by

$$p(x, y) = \frac{1}{864} e^{-x/24 - y/36}$$

for $x, y \geqslant 0$. What is the probability that both sensors fail within the first two years?

§15.6—Change of Variables

When we changed from Cartesian coordinates to polar coordinates in a double integral, we found that the area element $dx\,dy$ was replaced by $r\,dr\,d\theta$. In general, if we introduce new variables u and v via some mapping $(x, y) = \Phi(u, v)$, we must replace $dx\,dy$ by $|\mathrm{Jac}(\Phi)|\,du\,dv$, where

$$\mathrm{Jac}(\Phi) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

The determinant $\mathrm{Jac}(\Phi)$ is called the Jacobian of the transformation. Note that in the one-variable case, the Jacobian is just dx/du and the change of variables corresponds to an ordinary u-substitution.

Example 1. Find $\mathrm{Jac}(\Phi)$ for the mapping $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$ that relates polar to rectangular coordinates.

Example 2. Use the map $\Phi(u, v) = (4u + 3v, u + 2v)$ to evaluate $\displaystyle\iint_{\mathcal{D}} e^{2x+y}\,dA$, where $\mathcal{D}$ is the parallelogram spanned by the vectors $\langle 4, 1\rangle$ and $\langle 3, 2\rangle$.

The situation of Example 2 gives an indication of why the determinant $|\mathrm{Jac}(\Phi)|$ is the right correction factor to use in general. Notice that the area of the parallelogram can be computed using the cross product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 0 \\ 3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{k} \implies \text{Area} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 5 = |\mathrm{Jac}(\Phi)|.$$

Since the corresponding box $[0,1] \times [0,1]$ in the uv-plane has area 1, it follows that the change of variables shrinks the associated area elements by a factor of 5; multiplication by this factor thus gives the necessary correction.

Using linear approximation, we see that in general a small box of area $\Delta u\, \Delta v$ near the point (u,v) is essentially mapped to a parallelogram spanned by the vectors

$$\Phi(u+\Delta u, v) - \Phi(u,v) \approx \langle \partial x/\partial u, \partial y/\partial u \rangle \Delta u \quad \text{and} \quad \Phi(u, v+\Delta v) - \Phi(u,v) \approx \langle \partial x/\partial v, \partial y/\partial v \rangle \Delta v,$$

the area of which is $|\mathrm{Jac}(\Phi)|\Delta u \Delta v$. Thus we replace $dx\, dy$ by $|\mathrm{Jac}(\Phi)|\, du\, dv$.

The strategy of Example 2 can be generalized to integrate over the parallelogram $\mathcal{D}$ spanned by the vectors $\langle a, b \rangle$ and $\langle c, d \rangle$. We simply observe that the function $\Phi(u,v) = (au + cv, bu + dv)$ maps the unit square $[0,1] \times [0,1]$ in the uv-plane to $\mathcal{D}$ in the xy-plane.

Example 3. Evaluate $\displaystyle\iint_{\mathcal{D}} (x+y)\, dA$, where $\mathcal{D}$ is the parallelogram spanned by the vectors $\langle -2, 5 \rangle$ and $\langle 1, 7 \rangle$.

Example 4. Use the map $\Phi(u,v) = (au, bv)$ to find the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example 5. Use the change of variables $x = uv^{-1}$, $y = uv$ to evaluate $\displaystyle\iint_{\mathcal{D}} (x^2 + y^2)\, dA$, where $\mathcal{D}$ is the region in the first quadrant bounded by the lines $y = x$ and $y = 4x$ and the hyperbolas $xy = 1$ and $xy = 9$.

Sometimes it is more convenient to work with the inverse map $\Phi^{-1}(x, y) = (u, v)$. An extremely useful fact, which can be checked using the Chain Rule, is that

$$\operatorname{Jac}(\Phi^{-1}) = \frac{1}{\operatorname{Jac}(\Phi)}.$$

This often allows us to proceed without finding Φ explicitly.

Example 6. Let $\mathcal{D}$ be the triangular region in the xy-plane bounded by the lines $x - y = 1$, $x + y = 3$, and $y = 0$. Evaluate

$$\iint_{\mathcal{D}} \sqrt{\frac{x+y}{x-y}}\, dA.$$

An analogous result holds in three dimensions. When using coordinate transformation $(x, y, z) = \Phi(u, v, w)$, the volume element $dx\, dy\, dz$ is replaced by $|\operatorname{Jac}(\Phi)|\, du\, dv\, dw$, where

$$\operatorname{Jac}(\Phi) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}.$$

Note that this determinant can be viewed as a scalar triple product, which gives the volume of the parallelepiped determined by the gradient vectors of the transformation. This can be used to verify the claims of §15.4 concerning the conversions from rectangular to cylindrical and spherical coordinates (see problems 42 and 43).

§16.1—Vector Fields

A three-dimensional **vector field** is a function

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

that assigns a vector to each point in space. A planar (or two-dimensional) vector field has the simpler form $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$. Physically, vector fields are often used to represent gravitational or magnetic force at various points in space or the velocity vectors at various points in a fluid.

Example 1. Sketch the planar vector field $\mathbf{F}(x, y) = 2\mathbf{i} + x\mathbf{j}$.

A simple example of a vector field is the **gradient field** of a differentiable function $V(x, y, z)$,

$$\mathbf{F}(x, y, z) = \nabla V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k},$$

which maps each point to its gradient vector. In this case the function $V(x, y, z)$ is called a **potential** for $\mathbf{F}$, and we say that $\mathbf{F}$ is **conservative**.

Example 2. Let $V(x, y) = e^{x+3y} + \ln(1 + x^2 + y^4)$.

(a) Find the gradient field $\mathbf{F} = \nabla V$.

(b) Verify directly that the cross-partials $\partial F_1/\partial y$ and $\partial F_2/\partial x$ are equal for the vector field $\mathbf{F}$ from part (a).

Recall from §14.3 that $V_{xy} = V_{yx}$. Therefore, since $\mathbf{F}_1 = V_x$ and $\mathbf{F}_2 = V_y$, we see that $\partial F_1/\partial y = \partial F_2/\partial x$. Similarly, by applying the conditions $V_{xz} = V_{zx}$ and $V_{yz} = V_{zy}$, we get the following test:

The Cross-Partials Test. If the vector field

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

is conservative then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \text{and} \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

It turns out that the converse is true as well, and we will see in §16.3 how to use the cross-partials condition construct the potential function associated to a conservative vector field. In simple cases, one can often find the potential by inspection.

Example 3. Determine whether each of the following vector fields is conservative. If the field is conservative, find a potential function.

(a) $\mathbf{F}(x, y) = 2\mathbf{i} + x\mathbf{j}$

(b) $\mathbf{F}(x, y) = 2x\mathbf{i} + 3y^2\mathbf{j}$

(c) $F(x, y) = y\mathbf{i} + x\mathbf{j}$

(d) $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(e) $\mathbf{F} = 2yz\mathbf{i} + 3xz\mathbf{j} + 4xy\mathbf{k}$

(f) $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$

§16.2—Line Integrals

Scalar Line Integrals. Suppose we have a smooth curve $\mathcal{C}$ parametrized by $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leqslant t \leqslant b$. When t increases by a small amount dt, the resulting distance traveled along the curve is approximately $ds = \|\mathbf{c}'(t)\|dt$. We can therefore integrate a function $f(x, y, z)$ with respect to arc length along the curve $\mathcal{C}$ as follows:

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt.$$

This type of integral is known as a (scalar) line integral.

Example 1. Evaluate $\displaystyle\int_{\mathcal{C}} (x - y + z - 2)\, ds$, where $\mathcal{C}$ is the line segment from $(0, 1, 1)$ to $(1, 0, 1)$.

Line integrals can be used to calculate total mass and center of mass for objects lying along smooth curves in space, thus generalizing the formulas for one-dimensional rods.

Example 2. Find the mass of a wire of density $\rho(x, y, z) = 15\sqrt{y + 2}$ that lies along the curve

$$\mathbf{c}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k} \qquad (-1 \leqslant t \leqslant 1).$$

Vector Line Integrals and Work. Suppose that the vector field $\mathbf{F}(x, y, z)$ is a force function, such as a gravitational or electromagnetic field, and $\mathcal{C}$ is a smooth curve parametrized by $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leqslant t \leqslant b$. The component of $\mathbf{F}$ in the direction of motion along $\mathcal{C}$ is given by $\mathbf{F} \cdot \mathbf{T}$, where $\mathbf{T} = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ is the unit tangent vector to $\mathcal{C}$, and the work done by $\mathbf{F}$ over $\mathcal{C}$ is therefore given by

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

This formula can in fact be used to define the line integral of any vector field over a smooth curve.

Example 3. Calculate the work done by the force $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ along the path $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$ $(0 \leqslant t \leqslant 1)$.

Example 4. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = e^{y-x}\mathbf{i} + e^{2x}\mathbf{j}$ and $\mathcal{C}$ is the piecewise linear path from $(1, 1)$ to $(2, 2)$ to $(0, 2)$.

§16.3—Conservative Vector Fields

Recall that a vector field

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

is conservative on a given domain $\mathcal{D}$ if and only if there is a potential function V such that $\nabla V = \mathbf{F}$. When a vector field has a potential function, the potential essentially plays the role of an anti-derivative, and line integrals are calculated very easily using the following analogue of the Fundamental Theorem of Calculus.

Fundamental Theorem for Conservative Vector Fields: If $\mathbf{F} = \nabla V$ on some domain $\mathcal{D}$, then for every oriented curve $\mathcal{C}$ in $\mathcal{D}$ with initial point P and terminal point Q, one has

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = V(Q) - V(P).$$

In particular, if $\mathcal{C}$ is a *closed* curve (so that $P = Q$) then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

As a consequence, conservative vector fields have the following **path independence** property:

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$$

for all paths $\mathcal{C}_1$ and $\mathcal{C}_2$ with the same starting point and the same ending points. In the case of a gravitational or electromagnetic field, this means that the work done by $\mathbf{F}$ in moving from P to Q is the same for all paths from P to Q.

Example 1. Find the work done by the force $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ along a path from $(1, 3, 2)$ to $(2, 4, 5)$.

Recall from §16.1 that if the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is conservative then all pairs of cross-partials are equal. When working on a a simply connected domain $\mathcal{D}$ (that is, a region with no holes), the converse is also true. That is, $\mathbf{F}$ is conservative **if and only if**

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \text{and} \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

In simple cases such as Example 1 above, we can often find a potential for a conservative vector field by inspection. The following examples illustrate how to use the cross-partials criterion to construct a potential in less obvious situations.

Example 2. Find a potential function for the vector field $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$.

Example 3. Let $\mathbf{F} = (y-z)\mathbf{i} + (x+2yz-3z)\mathbf{j} + (y^2-x-3y+4)\mathbf{k}$. Evaluate $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathcal{C}$ is any path from $(0,0,0)$ to $(1,2,3)$.

§16.4—Parametrized Surfaces and Surface Integrals

While curves can be described using a function $\mathbf{c}(t)$ depending on a single parameter, surfaces require two parameters. In order to integrate over a surface, we need to find a vector-valued function of two variables, $G(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, that generates the points on the surface. For instance, a cylinder of radius R oriented along the z-axis can be parametrized by

$$G(\theta, z) = \langle R\cos\theta, R\sin\theta, z\rangle \quad (0 \leqslant \theta < 2\pi,\ -\infty < z < \infty).$$

Example 1. Consider the portion of the cylinder $x^2 + y^2 = 9$ with $0 \leqslant z \leqslant 5$.

(a) Find a parametrization $G(\theta, z)$ of the cylinder, and describe the *grid curves* $G(\theta, 2)$ and $G(\pi/4, z)$.

(b) Find the tangent vectors to the above grid curves, $\mathbf{T}_\theta = \partial G/\partial\theta$ and $\mathbf{T}_z = \partial G/\partial z$, at the point $(\theta_0, z_0) = (\pi/4, 2)$.

(c) Compute the normal vector $\mathbf{n}(\theta, z) = \mathbf{T}_\theta \times \mathbf{T}_z$ to the surface at the point $(\theta_0, z_0) = (\pi/4, 2)$.

The ideas in Example 1 can be generalized to define integration over a surface $\mathcal{S}$ parametrized by a function $G(u, v)$, with (u, v) lying in some *parameter domain* $\mathcal{D}$. By linear approximation, a rectangle of area $\Delta u\Delta v$ near a given point in the uv-plane is mapped by G to a "curved" parallelogram on the surface, which can be approximated by the parallelogram spanned by the vectors $\mathbf{T}_u\Delta u$ and $\mathbf{T}_v\Delta v$. Moreover, we know from §12.4 that the area of this parallelogram is

$$\|\mathbf{n}(u, v)\|\Delta u\Delta v = \|\mathbf{T}_u \times \mathbf{T}_v\|\Delta u\Delta v.$$

Thus we can integrate over a surface by integrating over the parameter domain, provided we replace the area element $du\,dv$ by $\|\mathbf{n}(u, v)\|\,du\,dv$:

$$\iint_{\mathcal{S}} f(x, y, z)\,dS = \iint_{\mathcal{D}} f(G(u, v))\|\mathbf{n}(u, v)\|\,du\,dv.$$

Note that this is analogous to the definition of scalar line integrals in §16.2. By taking $f(x, y, z) = 1$ we obtain the surface area of $\mathcal{S}$, whereas if f represents a density function then the surface integral calculates total mass or amount.

Example 2. Let $\mathcal{S}$ denote the truncated cylinder from Example 1.

(a) Calculate $\|\mathbf{n}(\theta, z)\|$.

(b) Find the surface area of the cylinder.

(c) If the cylinder is charged with density $\delta(x, y, z) = x^2 z$ coulombs per square unit, find the total charge on the cylinder.

Example 3. Let $\mathcal{S}$ denote the portion of the cone $z^2 = x^2 + y^2$ with $0 \leqslant z \leqslant 2$.

(a) Find a parametrization $G(r, \theta)$ for the truncated cone $\mathcal{S}$, and calculate $\|\mathbf{n}(r, \theta)\|$.

(b) A bacteria colony forms on the surface of $\mathcal{S}$, and its density is $\delta(x, y, z) = |xyz|$ million bacteria per square unit. Find the total number of bacteria on the cone.

Example 4. Evaluate $\displaystyle\iint_{\mathcal{S}} \sqrt{x^2+y^2}\, dS$, where $\mathcal{S}$ is the helicoid surface parametrized by

$$G(u,v) = \langle u\cos v, u\sin v, v\rangle \qquad (0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 2).$$

Example 5. Evaluate $\displaystyle\iint_{\mathcal{S}} (z-y)e^{2y}\, dS$, where $\mathcal{S}$ is defined by $z = x^3 + y$ with $x, y \in [0,1]$.

Note that a surface defined by a function $z = f(x,y)$ can always be parametrized by

$$G(x,y) = \langle x, y, f(x,y)\rangle,$$

and in this case one has $\mathbf{n}(x,y) = \langle -f_x, -f_y, 1\rangle$, and the surface area element is given by

$$dS = \|\mathbf{n}(x,y)\| dx\, dy = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

Note here the similarity with the arc length element $ds = \sqrt{1+f_x^2}\, dx$ for a curve $y = f(x)$.

§16.5—Surface Integrals of Vector Fields

We define the surface integral of a vector field $\mathbf{F}$ over a surface $\mathcal{S}$ parametrized by $G(u,v)$ with $(u,v) \in \mathcal{D}$ by

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(u,v)) \cdot \mathbf{n}(u,v)\, du\, dv.$$

A few comments are in order regarding this definition.

1. Here there are two possible orientations (outward or inward pointing) for the normal vector $\mathbf{n}(u,v)$. The orientation intended for $\mathcal{S}$ must be specified, since this affects the sign of the answer.

2. Note that the definitions of scalar and vector surface integrals mirror the corresponding definitions for line integrals, with the normal vector $\mathbf{n}(u,v)$ playing the role of $\mathbf{c}'(t)$.

3. The vector surface integral computes the **flux** of $\mathbf{F}$ across $\mathcal{S}$ by picking off the component of $\mathbf{F}$ in the direction normal to the surface at each point. When $\mathbf{F}$ is the velocity field of a fluid, the flux may be interpreted as the rate of fluid flow (volume per unit time) across the surface.

Example 1. Calculate $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle 0, y, x \rangle$ and $\mathcal{S}$ is the surface parametrized by

$$G(u,v) = \langle u^2, v, u^3 - v^2 \rangle \qquad (0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1),$$

oriented by upward-pointing normal vectors.

We emphasize again that the calculation of both scalar and vector surface integrals over a parametrized surface $G(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ depend on the normal vector obtained by crossing the derivative (tangent) vectors to the grid curves:

$$\mathbf{n}(u,v) = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}.$$

Example 2. Let $\mathcal{S}$ be the truncated paraboloid defined by $z = x^2 + y^2$ and $1 \leqslant z \leqslant 4$, oriented with outward-pointing normal vectors.

(a) Find a parametrization $G(r,\theta)$ of $\mathcal{S}$ using polar coordinates, and calculate the outward normal vector $\mathbf{n}(r,\theta)$.

(b) Compute the surface area of $\mathcal{S}$.

(c) Calculate the flux of the vector field $\mathbf{F} = \langle y, -x, z^2\rangle$ across $\mathcal{S}$.

Example 3. Let $\mathcal{S}$ be the hemisphere defined by $x^2 + y^2 + z^2 = 4$ and $z \geqslant 0$, oriented with outward-pointing normal vectors.

(a) Find a parametrization of $\mathcal{S}$, and obtain a formula for the outward normal vector.

(b) Calculate the flux of $\mathbf{F} = \langle 0, 0, x^2 \rangle$ across $\mathcal{S}$.

§17.1—Green's Theorem

Circulation. Suppose that $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ represents the velocity field of a fluid in two dimensions. In this case, $\mathbf{F} \cdot \mathbf{T}$ is the component of the velocity in the direction of the curve $\mathcal{C}$, so the analogue of the line integral we used for work in §16.2 now measures the *flow* along $\mathcal{C}$. When $\mathcal{C}$ is a closed loop, this is also called **circulation**:

$$\text{Circulation } = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy.$$

The notation on the right is a frequent shorthand for line integrals using differential forms. This arises from writing $\mathbf{F} = \langle F_1, F_2 \rangle$ and $\mathbf{s} = \langle x, y \rangle$, so that $d\mathbf{s} = \langle dx, dy \rangle$.

Example 1. Let $\mathbf{F}(x, y) = 4x\mathbf{i} + (x - y)\mathbf{j}$, and let $\mathcal{C}$ be the circle

$$\mathbf{c}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} \quad (0 \leqslant t \leqslant 2\pi).$$

Find the circulation of $\mathbf{F}$ around $\mathcal{C}$.

The following theorem often simplifies the calculation of a circulation line integral by relating it to a double integral over a region in the plane. A **simple** curve is one that does not intersect itself except possibly at the starting and ending points.

Green's Theorem for Circulation. If $\mathcal{C}$ is a simple closed curve traversed counter-clockwise and $\mathcal{D}$ is the region enclosed by $\mathcal{C}$, then

$$\oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA,$$

provided that F_1 and F_2 are differentiable functions with continuous first partial derivatives. Notice that if $\mathbf{F}$ is conservative, then the cross-partials test shows that the integral on the right is zero, which also follows from path independence.

In Example 1, we have $F_1 = 4x$ and $F_2 = x - y$, so $\partial F_2/\partial x - \partial F_1/\partial y = 1$, and Green's Theorem immediately gives

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} 1 \cdot dA = \text{Area}(\mathcal{D}) = \pi a^2.$$

Note: Green's Theorem is frequently stated in terms of the domain $\mathcal{D}$ and its boundary curve, denoted $\partial\mathcal{D}$. In this instance, the references to $\mathcal{C}$ in the theorem are replaced by $\partial\mathcal{D}$.

Example 2. Use Green's Theorem to evaluate the integral $\oint_{\mathcal{C}} 3y\,dx + 2x\,dy$, where $\mathcal{C}$ is the boundary of the region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \sin x$, oriented counter-clockwise.

Example 3. Let $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (x + y^2)\mathbf{j}$, and let $\mathcal{C}$ be the square bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, oriented counter-clockwise. Use Green's Theorem to find the circulation of $\mathbf{F}$ around $\mathcal{C}$.

Example 4. Let $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$, and let $\mathcal{C}$ be the triangle bounded by $y = 0$, $x = 1$, and $y = x$, oriented counter-clockwise. Use Green's Theorem to find the circulation of $\mathbf{F}$ around $\mathcal{C}$.

§17.2—Stokes' Theorem

The quantity $\partial F_2/\partial x - \partial F_1/\partial y$ occurring in Green's Theorem is the $\mathbf{k}$ component of a vector called the curl, associated to the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$:

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Here we view ∇ as an operator: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

Example 1. Calculate the curl of the vector field $\mathbf{F} = \langle xy^2, e^{2x}, y^2 + 3z \rangle$.

Stokes' Theorem. For nicely behaved oriented surfaces, we have the following extension of Green's Theorem:

$$\oint_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\partial\mathcal{S}$ denotes the boundary of $\mathcal{S}$, oriented so that a normal vector "walking" along the curve has the surface on its left. In words, the theorem states that the circulation along the boundary is equal to the flux of the curl through the surface.

Remarks:

1. Notice that if the surface $\mathcal{S}$ is a region $\mathcal{D}$ in the xy-plane and $\mathbf{F} = \langle F_1, F_2 \rangle$ is a two-dimensional vector field, then $\mathcal{S}$ can be parametrized by $G(x, y) = \langle x, y, 0 \rangle$, so that $\mathbf{n}(x, y) = \mathbf{k}$. Thus $\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}(x, y) = \partial F_2/\partial x - \partial F_1/\partial y$, so Stokes' Theorem reduces to Green's Theorem.

2. If the vector field $\mathbf{F}$ is conservative, then the cross-partials test shows that $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, and Stokes' Theorem recovers the path-independence property for line integrals by showing that the circulation along the boundary is zero.

3. For a surface with no boundary, such as a sphere, we regard the boundary curve $\partial\mathcal{S}$ as the empty set, and the associated line integral is zero.

We illustrate Stokes' Theorem with the following example, in which we calculate line integral over the boundary and the surface integral of the curl separately and show that they are equal. In practice, the value of the theorem allows us to work with whichever integral is easier to handle.

Example 2. Let $\mathcal{S}$ denote the hemisphere defined by $x^2 + y^2 + z^2 = 4$ and $z \geqslant 0$, oriented with outward-pointing normal vectors, and let $\mathbf{F} = \langle -y, x, x+z \rangle$. Recall from Example 3 in the §16.5 notes that $\mathcal{S}$ has parametrization

$$G(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, \cos\phi \rangle \qquad (0 \leqslant \phi \leqslant \pi/2,\ 0 \leqslant \theta \leqslant 2\pi)$$

and outward normal vector

$$\mathbf{n}(\phi, \theta) = 4\sin\phi\,\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle.$$

(a) Find a parametrization of the boundary curve $\partial\mathcal{S}$, and use it to calculate $\displaystyle\oint_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s}$.

(b) Calculate $\displaystyle\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

Since a single curve can be the boundary for many different surfaces, Stokes' Theorem allows us to calculate a circulation line integral by integrating over a convenient surface of our choice. Likewise, a surface integral of a curl depends only on the boundary curve, so if $\mathbf{F} = \text{curl}(\mathbf{A})$, the flux of $\mathbf{F}$ is **surface-independent** and can be calculated as a line integral of $\mathbf{A}$ over the boundary. In the latter case, $\mathbf{A}$ is called a **vector potential** for $\mathbf{F}$. We explore these uses of Stokes' Theorem in the next two examples.

Example 3. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \langle -y^2, x, z^2 \rangle$ and $\mathcal{C}$ is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented clockwise when viewed from above.

Example 4. Let $\mathbf{F} = \text{curl}(\mathbf{A})$, where $\mathbf{A} = \langle y + z, \sin(xy), e^{xyz} \rangle$. Calculate the flux of $\mathbf{F}$ through the surface $\mathcal{S}$ shown in the diagram, whose boundary is the unit circle $x^2 + z^2 = 1$ in the xz-plane, oriented as shown.

§17.3—The Divergence Theorem

The **divergence** of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is defined by

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note that while $\operatorname{curl}(\mathbf{F})$ is a vector, $\operatorname{div}(\mathbf{F})$ is a scalar.

Example 1. Find the divergence of the vector field $\mathbf{F} = \langle e^{x^2 y} + z, xy^4 z, \sin(xy) + \ln z \rangle$.

Our final theorem relates the flux of a vector field across a surface to a triple integral of the divergence over the enclosed region. This is in keeping with the general theme of relating the integral of a function on a boundary to the integral of a derivative over the interior. The fundamental theorem of calculus, Green's Theorem, and Stokes' Theorem all have this type of structure.

The Divergence Theorem. Let $\mathcal{S}$ be a closed surface, oriented with outward-pointing normal vectors, that encloses a region $\mathcal{W}$ in $\mathbf{R}^3$. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV,$$

provided that all points in $\mathcal{W}$ lie in the domain of $\mathbf{F}$.

Example 2. Use the Divergence Theorem to calculate the outward flux of $\mathbf{F} = \langle 2x, 3y, 5z \rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

The Divergence Theorem allows us to interpret div($\mathbf{F}$) as **flux per unit volume**, while Stokes' Theorem shows that curl($\mathbf{F}$) can be viewed as **circulation per unit area**. Below we verify the Divergence Theorem in a particular case by computing the integrals on both sides independently.

Example 3. Let $\mathcal{S}$ denote the closed cylinder of radius 2 and height 5 whose base is the disk $x^2 + y^2 \leqslant 4$ in the xy-plane, and let $\mathbf{F} = \langle y, yz, z^2 \rangle$.

(a) Calculate $\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV$, where $\mathcal{W}$ is the region enclosed by $\mathcal{S}$.

(b) Calculate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ directly by integrating over the top, bottom, and sides of the cylinder.

An important restriction on applying the Divergence Theorem is that the domain of $\mathbf{F}$ contain the region $\mathcal{W}$. We explore the consequences of this subtlety for inverse-square fields below.

Example 4. Consider the electric field $\mathbf{E} = \dfrac{1}{\rho^2}\mathbf{e}$, where

$$\mathbf{e} = \frac{1}{\rho}\langle x, y, z\rangle = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle.$$

(a) Show that $\operatorname{div}(\mathbf{E}) = 0$.

(b) Calculate the outward flux of $\mathbf{E}$ through the sphere $x^2 + y^2 + (z-3)^2 = 1$.

(c) Calculate the outward flux of $\mathbf{E}$ through the sphere $x^2 + y^2 + z^2 = 1$.