MAT 162—Exam #2—10/25/11

Name: Solutions

Show all work using correct mathematical notation. Calculators are not permitted.

1. (10 points) Evaluate $\int \frac{2x+5}{x^2-3x+2}\,dx$.

$$\frac{2x+5}{x^2-3x+2} = \frac{2x+5}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\Rightarrow \quad 2x+5 = A(x-2) + B(x-1)$$

$$x=1: \quad 7 = -A \quad \Rightarrow \quad A = -7$$

$$x=2: \quad 9 = B$$

Thus $\int \frac{2x+5}{x^2-3x+2}\,dx = 9\int \frac{dx}{x-2} - 7\int \frac{dx}{x-1}$

$$= 9\ln|x-2| - 7\ln|x-1| + C$$

2. (15 points) Evaluate $\int x^2 e^{3x}\,dx$.

①

$u = x^2 \qquad dv = e^{3x}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{3}e^{3x}$

②

$u = x \qquad dv = e^{3x}\,dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\,dx$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx\right)$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

3. (10 points) Evaluate $\int \frac{2x+1}{x^2+9}\,dx$. *Hint:* Split the integral into two parts.

$$\int \frac{2x+1}{x^2+9}\,dx = \int \frac{2x\,dx}{x^2+9} + \int \frac{dx}{x^2+9}$$

$u = x^2+9$

$du = 2x\,dx$

$$= \int \frac{du}{u} + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

$$= \ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

4. (15 points) Evaluate $\int \sin^2 x \cos^5 x\,dx$.

$$\int \sin^2 x \cos^5 x\,dx = \int \sin^2 x\,(\cos^2 x)^2 \cos x\,dx$$

$$= \int \sin^2 x\,(1-\sin^2 x)^2 \cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int u^2(1-u^2)^2\,du$$

$$= \int u^2(1-2u^2+u^4)\,du$$

$$= \int (u^2-2u^4+u^6)\,du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

5. (10 points) Evaluate the improper integral $\int_0^4 \frac{1}{\sqrt{x}}\,dx$, or show that it diverges.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^4 x^{-1/2}\,dx$$

$$= \lim_{a\to 0^+} 2x^{1/2}\Big|_a^4$$

$$= \lim_{a\to 0^+} \left(2\sqrt{4} - 2\sqrt{a}\right)$$

$$= 4$$

6. (15 points) Consider the integral $\int_3^5 \ln x\,dx$.

(a) Use Simpson's Rule with $N = 4$ to approximate the integral. Just write out the terms in your sum–do not attempt to add them up.

$$\Delta x = \frac{5-3}{4} = \frac{1}{2}$$

$$S_4 = \frac{1}{6}\left(\ln 3 + 4\ln 3.5 + 2\ln 4 + 4\ln 4.5 + \ln 5\right)$$

(b) Determine the maximum possible error in your estimate from part (a). You may leave your answer in messy form–do not attempt to do any complicated arithmetic.

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{6}{x^4}$

$$|f^{(4)}(x)| = \frac{6}{x^4} \leq \frac{6}{3^4} = \frac{2}{27} \quad \text{on } [3,5]$$

Thus $$|E_S| \leq \frac{\frac{2}{27}(5-3)^5}{180\cdot 4^4}$$

$$= \frac{1}{27\cdot 720}$$

$$= \frac{1}{19{,}440}$$

7. (18 points) Evaluate $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$.

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$\sqrt{4+x^2} = \sqrt{4(1+\tan^2\theta)}$

$= 2\sec\theta$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta \cdot 2\sec^2\theta \, d\theta}{2\sec\theta}$$

$$= 8\int \tan^3\theta \sec\theta \, d\theta$$

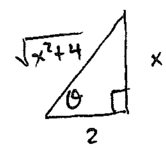
$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta \, d\theta$$

$u = \sec\theta$

$du = \sec\theta\tan\theta \, d\theta$

$$= 8\int (u^2-1)\,du$$

$$= 8\left(\tfrac{1}{3}u^3 - u\right) + C$$



$$= 8\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\cdot\frac{\sqrt{x^2+4}}{2} + C$$

$$= \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

8. (7 points) Consider the improper integral $\displaystyle\int_2^\infty \frac{x+x^2}{x^3-1}\,dx$.

(a) Does the integral converge or diverge?

Diverges

(b) Which of the following integrals can the original be compared with to reach the above conclusion? (Circle all that apply.)

(i) $\displaystyle\int_2^\infty \frac{1}{x^{2/3}}\,dx$ (ii) $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ (iii) $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ (iv) $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$

$$\frac{x+x^2}{x^3-1} \geq \frac{x^2}{x^3} = \frac{1}{x} \qquad p=1$$