MAT 162—Exam #1—9/26/13

Name: Solutions

Show all work using correct mathematical notation. Calculators are not allowed.

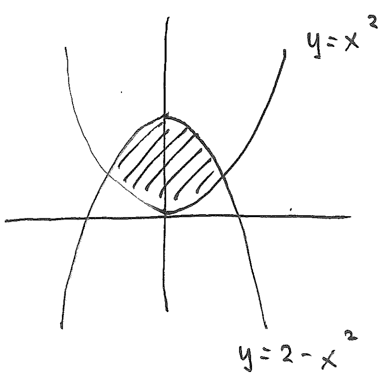
1. (10 points) Find the average value of the function $f(x) = e^{2x}$ on the interval $[0, 3]$.

$$f_{ave} = \frac{1}{3-0}\int_0^3 e^{2x}\,dx$$

$$= \frac{1}{3}\cdot\frac{1}{2}e^{2x}\Big|_0^3$$

$$= \frac{1}{6}(e^6 - 1)$$

2. (15 points) Sketch the region bounded by the parabolas $y = x^2$ and $y = 2 - x^2$, and find its area.



Int pts: $x^2 = 2 - x^2$

$\Rightarrow \quad 2x^2 = 2$

$\Rightarrow \quad x = \pm 1$

$$\text{Area} = \int_{-1}^{1} (2 - x^2 - x^2)\,dx$$

$$= 2\int_0^1 (2 - 2x^2)\,dx$$

$$= 2\left(2x - \frac{2}{3}x^3\right)\Big|_0^1$$

$$= 2\left(2 - \frac{2}{3}\right)$$

$$= \frac{8}{3}$$

3. (10 points) The mass density (in kg/m) of a rod lying along the interval $0 \leqslant x \leqslant 2$ is given by $\rho(x) = \dfrac{x^2}{x^3+1}$. Find the total mass of the rod.

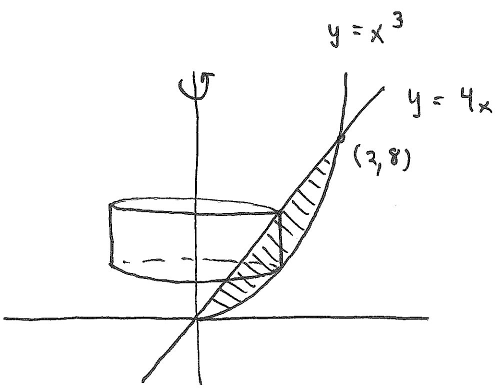
$$\text{Mass} = \int_0^2 \frac{x^2}{x^3+1}\,dx \qquad \text{Let } u = x^3+1, \quad du = 3x^2\,dx$$

$$= \frac{1}{3}\int_1^9 \frac{du}{u}$$

$$= \frac{1}{3}\ln|u|\Big|_1^9$$

$$= \frac{1}{3}\ln 9$$

4. (15 points) Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curves $y = x^3$ and $y = 4x$ about the y-axis. Include a sketch of the region, along with a representative disk, washer, or shell.



Shell radius : $r(x) = x$

Shell height : $h(x) = 4x - x^3$

$$\text{Volume} = \int_0^2 2\pi x\,(4x - x^3)\,dx$$

$$= 2\pi \int_0^2 (4x^2 - x^4)\,dx$$

$$= 2\pi\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$

$$= 2\pi\left(\frac{32}{3} - \frac{32}{5}\right)$$

$$= 64\pi \cdot \frac{2}{15}$$

$$= \frac{128\pi}{15}$$

Int pts :

$x^3 = 4x$

$\Rightarrow \; x(x^2 - 4) = 0$

$\Rightarrow \; x = 0, \pm 2$

5. (10 points) Consider the portion of the curve $y = 4/x$ between the points $(1, 4)$ and $(2, 2)$. Set up (**but do not evaluate**) integrals that give

(a) the length of the curve

$f(x) = 4/x$

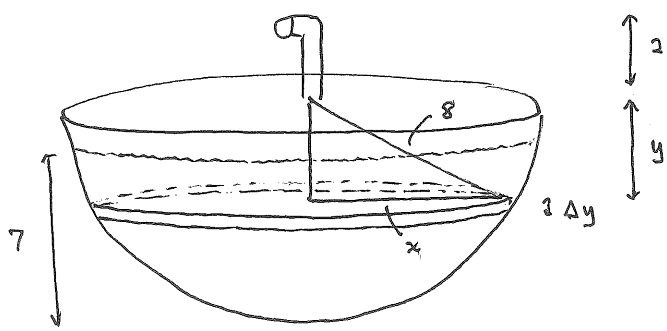
$\Rightarrow f'(x) = -4/x^2$

$$L = \int_1^2 \sqrt{1 + 16/x^4}\, dx$$

(b) the area of the surface obtained by revolving the curve about the x-axis

$$S = \int_1^2 2\pi \cdot \frac{4}{x} \cdot \sqrt{1 + 16/x^4}\, dx$$

6. (15 points) A hemispherical tank of radius 8 meters is filled to a height of 7 meters with water, which weighs 9800 N/m^3. Set up (**but do not evaluate**) a definite integral that gives the work required to pump all the water out through a spout that extends 2 meters above the tank's top. Show a representative slice and identify your variable(s) on the sketch provided.



Pythagorean Thm $\Rightarrow$

$$x^2 + y^2 = 8^2$$

$$\Rightarrow x^2 = 64 - y^2$$

Volume of slice $= \pi x^2 \Delta y$

$= \pi(64 - y^2)\, \Delta y$

Weight of slice $= 9800\pi(64 - y^2)\, \Delta y$

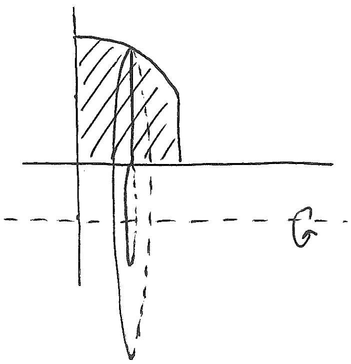
Distance moved by slice $= y + 2$

Work on slice $= 9800\pi(64 - y^2)(y+2)\, \Delta y$

Total Work $= \int_1^8 9800\pi(64 - y^2)(y+2)\, dy$

7. (15 points) Consider the region bounded by the curve $y = 1 + \cos x$ and the lines $y = 0$, $x = 0$, and $x = \pi/2$. Set up (**but do not evaluate**) definite integrals that give the volumes of the solids obtained by rotating the region about the following axes. For each part, show a representative disk, washer, or shell on the sketch provided.

(a) the line $y = -1$

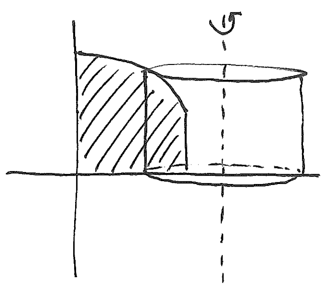


Washers : $R(x) = 2 + \cos x$

$r(x) = 1$

$$V = \int_0^{\pi/2} \pi \left[(2 + \cos x)^2 - 1 \right] dx$$

(b) the line $x = 2$

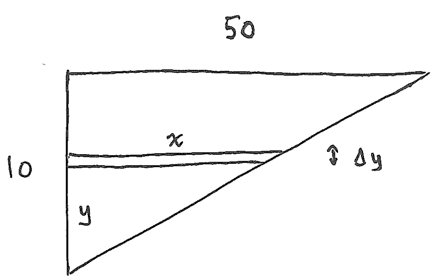


Shells : $r(x) = 2 - x$

$h(x) = 1 + \cos x$

$$V = \int_0^{\pi/2} 2\pi (2 - x)(1 + \cos x)\, dx$$

8. (10 points) A rectangular swimming pool is 50 feet long, 20 feet wide, 10 feet deep at one end, and zero depth at the other end. The pool is full of water, which weighs 62.4 lb/ft^3. Set up (**but do not evaluate**) a definite integral that gives the hydrostatic force on one of the triangular sides of the pool. Show a representative strip and identify your variable(s) on the sketch provided.



Pressure along strip $= 62.4\,(10 - y)$

Area of strip $= x\,\Delta y$

$= 5y\,\Delta y$

Force on strip $= 62.4\,(10 - y) \cdot 5y\,\Delta y$

Total Force $= \displaystyle\int_0^{10} 62.4\,(10 - y) \cdot 5y\, dy$

Similar Δ's $\Rightarrow$

$$\frac{x}{y} = \frac{50}{10}$$

$\Rightarrow x = 5y$