MAT 162—Exam #1—9/27/11

Name: Solutions

Show all work using correct mathematical notation. Calculators are not allowed.

1. (10 points) Find the average value of the function $f(x) = \dfrac{1}{x^3}$ on the interval $[1, 3]$.

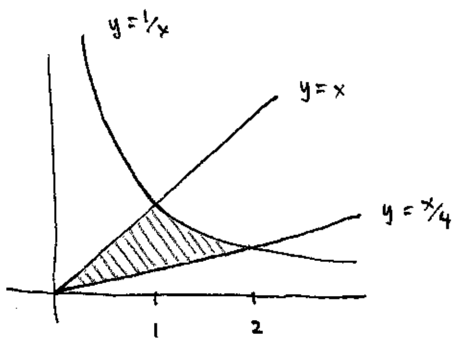
$$f_{ave} = \frac{1}{3-1}\int_1^3 \frac{1}{x^3}\,dx$$

$$= \frac{1}{2}\cdot\left(\frac{-1}{2x^2}\right)\Big|_1^3$$

$$= -\frac{1}{4}\left(\frac{1}{9}-1\right)$$

$$= \frac{2}{9}$$

2. (15 points) Find the area of the region enclosed by the curves $y = 1/x$, $y = x$, and $y = x/4$.



$$\text{Area} = \int_0^1 \left(x - \frac{x}{4}\right)dx + \int_1^2 \left(\frac{1}{x} - \frac{x}{4}\right)dx$$

$$= \frac{3}{8}x^2\Big|_0^1 + \left(\ln|x| - \frac{1}{8}x^2\right)\Big|_1^2$$

$$= \frac{3}{8} + \ln 2 - \frac{1}{2} - \left(\ln 1 - \frac{1}{8}\right)$$

$$= \ln 2$$

Int pts:

$$x = \frac{1}{x} \Rightarrow x^2 = 1$$

$$\Rightarrow x = 1$$

$$\frac{x}{4} = \frac{1}{x} \Rightarrow x^2 = 4$$

$$\Rightarrow x = 2$$

3. (10 points) The linear mass density (in kg/m) of a rod lying along the interval $0 \le x \le 5$ is given by $\rho(x) = e^{-2x}$. Find the total mass of the rod.

$$M = \int_0^5 e^{-2x}\,dx$$

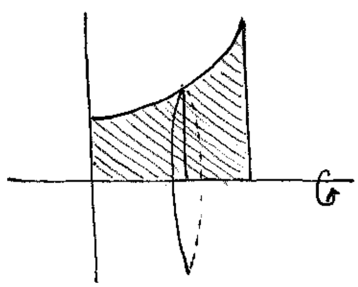
$$= -\frac{1}{2} e^{-2x} \Big|_0^5$$

$$= -\frac{1}{2}\left(e^{-10} - e^{0}\right)$$

$$= \frac{1}{2}\left(1 - e^{-10}\right)$$

4. (15 points) Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curve $y = x^3 + 1$ and the lines $y = 0$, $x = 0$, and $x = 1$ about the x-axis. Be sure to show a representative disk/washer or shell on the sketch provided.

Disk at x has radius $R(x) = x^3 + 1$.



$$V = \int_0^1 \pi (x^3+1)^2\,dx$$

$$= \pi \int_0^1 (x^6 + 2x^3 + 1)\,dx$$

$$= \pi \left(\frac{x^7}{7} + \frac{x^4}{2} + x\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{7} + \frac{1}{2} + 1\right)$$

$$= \frac{23\pi}{14}$$

5. (10 points) Set up (**but do not evaluate**) a definite integral that gives the surface area of the object obtained by rotating the portion of the curve $y = \sin 3x$ with $0 \le x \le \pi/3$ about the x-axis.

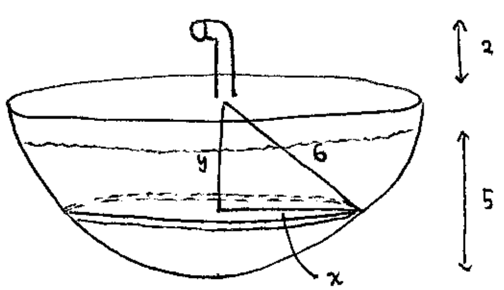
$$f(x) = \sin 3x$$

$$f'(x) = 3 \cos 3x$$

$$S = \int_0^{\pi/3} 2\pi \sin 3x \sqrt{1 + (3\cos 3x)^2}\, dx$$

$$= 2\pi \int_6^{\pi/3} \sin 3x \sqrt{1 + 9\cos^2 3x}\, dx$$

6. (15 points) A hemispherical tank of radius 6 meters is filled to a height of 5 meters with water, which weighs 9800 N/m^3. Set up (**but do not evaluate**) a definite integral that gives the work required to pump all the water out through a spout that extends 2 meters above the tank's top. Show a representative slice and identify your variable(s) on the sketch provided.



Volume of slice $= \pi x^2 \, \Delta y$

$= \pi (36 - y^2)\, \Delta y$

Weight of slice $= 9800\, \pi (36 - y^2)\, \Delta y$

Distance moved by slice $= y + 2$

Pythagorean Theorem

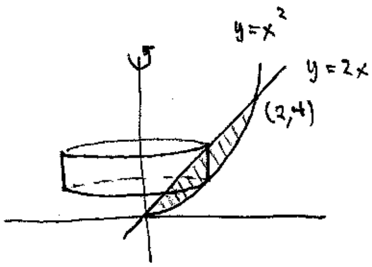
$\Rightarrow \; x^2 + y^2 = 36$

Work on slice $= 9800\, \pi (36 - y^2)(y+2)\, \Delta y$

Total Work $= \displaystyle\int_1^6 9800\, \pi (36 - y^2)(y+2)\, dy$

7. (15 points) Consider the region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$. Set up (**but do not evaluate**) definite integrals that give the volumes of the solids obtained by rotating the region about the following axes. For each part, include a sketch of the region and a representative disk/washer or shell.

(a) the y-axis

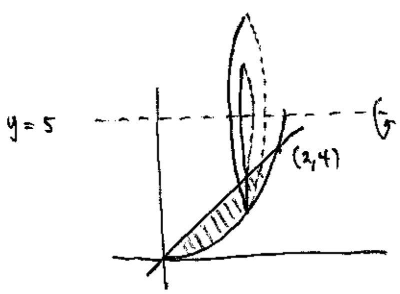


Shell radius : $r(x) = x$

Shell height : $h(x) = 2x - x^2$

$$V = \int_0^2 2\pi x \left(2x - x^2\right) dx$$

(b) the line $y = 5$



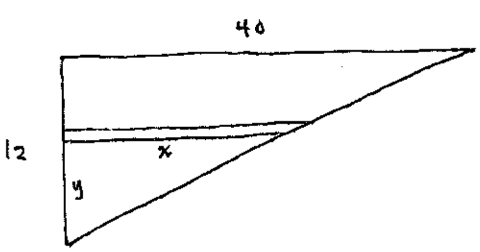
Washers :

Outer radius $R(x) = 5 - x^2$

Inner radius $r(x) = 5 - 2x$

$$V = \int_0^2 \pi \left[(5-x^2)^2 - (5-2x)^2 \right] dx$$

8. (10 points) A rectangular swimming pool is 40 feet long, 30 feet wide, 12 feet deep at one end, and zero depth at the other end. The pool is full of water, which weighs 62.4 lb/ft^3. Set up (**but do not evaluate**) a definite integral that gives the hydrostatic force on one of the triangular sides of the pool. Show a representative strip and identify your variable(s) on the sketch provided.



Pressure along strip $= 62.4\,(12 - y)$

Area of strip $= x\,\Delta y$

$= \frac{10}{3} y\,\Delta y$

Similar Δ's $\Rightarrow$

$\frac{x}{y} = \frac{40}{12}$

$\Rightarrow x = \frac{10}{3} y$

Force on strip $= 62.4\,(12 - y) \cdot \frac{10}{3} y\,\Delta y$

Total Force $= \int_0^{12} 62.4 \cdot \frac{10}{3} y\,(12 - y)\,dy$