MAT 162—West Chester University—Spring 2011
Notes on Rogawski's Calculus: Early Transcendentals
Scott Parsell

Review of the Definite Integral

To approximate the area between the graph of a continuous function $y = f(x)$ and the x-axis on the interval $[a, b]$, we divide into N subintervals of width $\Delta x = (b - a)/N$, and write $x_i = a + i\Delta x$ for the right-hand endpoint of the ith subinterval. Using a rectangle of height $f(x_i)$ and width Δx to approximate the area under the curve on the ith subinterval, we see that the **Riemann sum**

$$R_N = \sum_{i=1}^{N} f(x_i)\Delta x$$

approximates the total area under the curve on the interval $[a, b]$. We get the exact area by letting $N \to \infty$, which gives the **definite integral** of f from a to b:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x.$$

If f takes both positive and negative values, the definite interval gives the signed (or net) area under the curve. When $f(x)$ has an **anti-derivative**, the calculation of the definite integral is greatly simplified by the following theorem, which may be interpreted as saying that the definite integral of a rate of change gives the total change.

The Fundamental Theorem of Calculus (Part I). If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example 1. Evaluate each of the following definite integrals.

(a) $\displaystyle\int_0^4 \sqrt{x}\,dx$

$= \frac{2}{3} x^{3/2} \Big|_0^4$

$= \frac{2}{3}(4^{3/2} - 0)$

$= \frac{16}{3}$

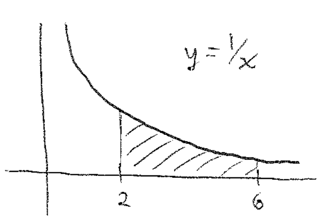
(b) $\displaystyle\int_0^{\pi/6} \cos 3x\,dx$ $\qquad u = 3x$, $du = 3\,dx$

$= \frac{1}{3}\int_0^{\pi/2} \cos u\,du$

$= \frac{1}{3} \sin u \Big|_0^{\pi/2}$

$= \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{3}$

Example 2. Find the area under the curve $y = 1/x$ between $x = 2$ and $x = 6$.



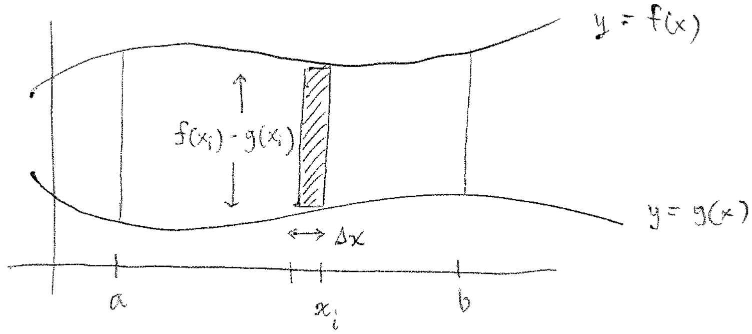
$A = \int_2^6 \frac{1}{x}\,dx$

$= \ln x \Big|_2^6 = \ln 6 - \ln 2$

$= \ln 3$

§6.1—Area Between Two Curves

Slicing the region. To find the area of the region bounded above by $y = f(x)$ and below by $y = g(x)$ on the interval $a \le x \le b$, we again divide the interval into N subintervals. If x_i denotes any point in the ith subinterval, then we can use a rectangle of height $f(x_i) - g(x_i)$ and width Δx to approximate the ith "slice" of our region:



We then add up the contributions from all the slices and let N tend to infinity to get an exact answer. Thus we have

$$\text{Area} = \lim_{N\to\infty} \sum_{i=1}^{N} (f(x_i) - g(x_i))\Delta x = \int_a^b (f(x) - g(x))\,dx.$$

Note that we're using the definition of the definite integral as a limit of Riemann sums. We can think of dx as an infinitesimally small version of Δx.

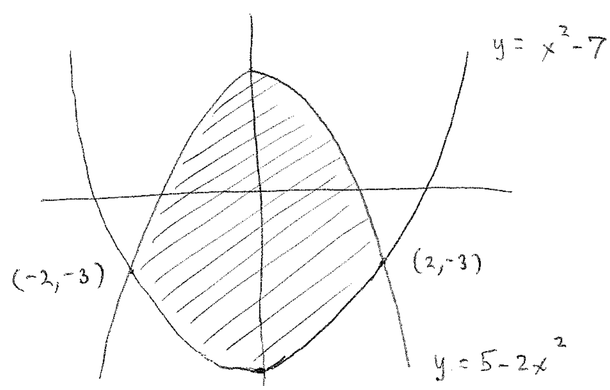
Example 1. Find the area of the region bounded by the curves $y = x^2 - 7$ and $y = 5 - 2x^2$.

Intersection points :

$x^2 - 7 = 5 - 2x^2$

$\Rightarrow \quad 3x^2 = 12$

$\Rightarrow \quad x = \pm 2$



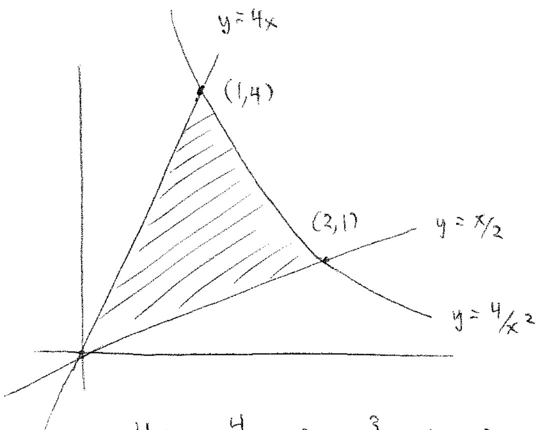
$$A = \int_{-2}^{2} \left(5 - 2x^2 - (x^2 - 7)\right) dx$$

$$= \int_{-2}^{2} (12 - 3x^2)\, dx$$

$$= 2\int_0^2 (12 - 3x^2)\, dx$$

$$= 2\left(12x - x^3\right)\Big|_0^2 = 2(24 - 8) = 32$$

Example 2. Find the area of the region bounded by the curve $y = 4/x^2$ and the lines $y = x/2$ and $y = 4x$.



$$A = \int_0^1 \left(4x - \frac{x}{2}\right) dx + \int_1^2 \left(\frac{4}{x^2} - \frac{x}{2}\right) dx$$

$$= 2x^2 - \frac{x^2}{4} \Big|_0^1 + \left(-\frac{4}{x} - \frac{x^2}{4}\right)\Big|_1^2$$

$$= 2 - \frac{1}{4} + (-2-1) - \left(-4 - \frac{1}{4}\right)$$

$$= 3$$

$$4x = \frac{4}{x^2} \Rightarrow x^3 = 1 \Rightarrow x = 1$$

$$\frac{x}{2} = \frac{4}{x^2} \Rightarrow x^3 = 8 \Rightarrow x = 2$$

Sometimes it is easier to calculate the area of a region by integrating with respect to y. In this case, the area is given by

$$\int_c^d [f(y) - g(y)]\, dy,$$

where $x = f(y)$ is the right-hand curve and $x = g(y)$ is the left-hand curve.

Example 3. Find the area of the region bounded by the line $y = x - 5$ and the parabola $y^2 = 2x - 2$.

$x = y + 5$

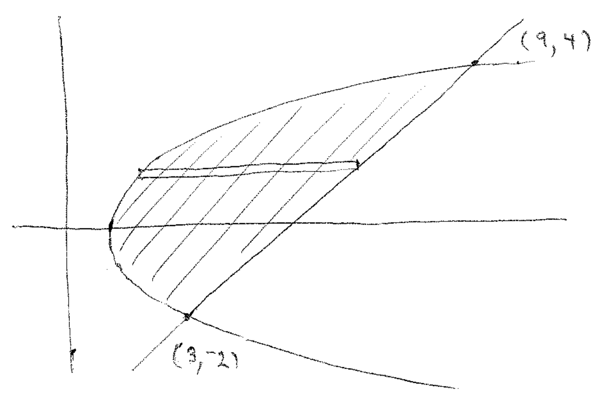
$x = \frac{1}{2}y^2 + 1$

$$\frac{1}{2}y^2 + 1 = y + 5$$

$$\Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y-4)(y+2) = 0$$

$$\Rightarrow y = 4, -2$$



$$A = \int_{-2}^{4} \left(y + 5 - \frac{1}{2}y^2 - 1\right) dy$$

$$= \frac{y^2}{2} + 4y - \frac{y^3}{6} \Big|_{-2}^{4}$$

$$= 8 + 16 - \frac{32}{3} - \left(2 - 8 + \frac{4}{3}\right)$$

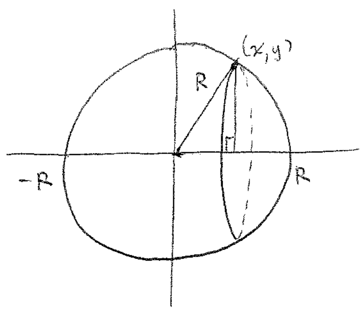
$$= 18$$

§6.2—Setting Up Integrals: Volume, Density, Average Value

Computing volumes by slicing. Suppose we have a 3-dimensional solid whose x-coordinates all lie in the interval $[a, b]$. Suppose we slice the solid perpendicular to the x-axis, and let $A(x)$ denote the area of the cross-section formed by this slice. If we slice our solid into N "slabs" of width Δx and let x_i be any x-coordinate occurring in the ith slab, then we can approximate the volume of this slab as $A(x_i)\Delta x$. Adding up the contributions from all the slices and taking the limit as $N \to \infty$, we get

$$\text{Volume} = \lim_{N\to\infty} \sum_{i=1}^{N} A(x_i)\Delta x = \int_a^b A(x)\,dx.$$

Example 1. Use slicing to find the volume of a sphere of radius R.



Cross-section at x is a circle of radius y, so

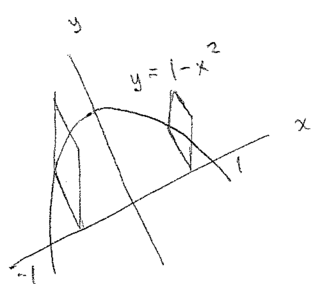
$$A(x) = \pi y^2 = \pi(R^2 - x^2).$$

Thus
$$\begin{aligned} V &= \int_{-R}^{R} \pi(R^2 - x^2)\,dx \\ &= 2\pi \int_0^R (R^2 - x^2)\,dx \\ &= 2\pi\left(R^2 x - \frac{x^3}{3}\right)\Big|_0^R \\ &= 2\pi\left(R^3 - \frac{1}{3}R^3\right) = \frac{4}{3}\pi R^3 \end{aligned}$$

Example 2. The base of a solid is the region bounded by the parabola $y = 1 - x^2$ and the x-axis, and its cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.

Cross-section at x is a square of radius y, so

$$A(x) = y^2 = (1 - x^2)^2.$$



Thus
$$\begin{aligned} V &= \int_{-1}^{1} (1 - x^2)^2\,dx \\ &= 2\int_0^1 (1 - 2x^2 + x^4)\,dx \\ &= 2\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 \\ &= 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) \\ &= \frac{16}{15} \end{aligned}$$

Density. To find the mass of a thin wire or rod of variable density, we slice it into small pieces. Suppose that the rod's linear mass density at x (say in kg/m) is $\rho(x)$ and that the rod extends from $x = a$ to $x = b$. Then the mass of a small piece of the rod of length Δx centered at x_i is given by $M_i \approx \rho(x_i)\Delta x$, so the total mass is

$$M = \lim_{N\to\infty} \sum_{i=1}^{N} M_i = \lim_{N\to\infty} \sum_{i=1}^{N} \rho(x_i)\,\Delta x = \int_a^b \rho(x)\,dx.$$

Example 3. Find the total mass of a 1-meter rod whose linear density function is $\rho(x) = 10(x+1)^{-3}$ kg/m for $0 \le x \le 1$.

$$M = \int_0^1 10(x+1)^{-3}\,dx = 10\int_1^2 u^{-3}\,du$$

$u = x+1$

$du = dx$

$$= -5u^{-2}\Big|_1^2$$

$$= -5\left(\frac{1}{4} - 1\right)$$

$$= \frac{15}{4} \text{ kg}$$

A similar idea can be used to analyze two-dimensional mass or population densities that vary with the distance from a central point. For instance, to find the population within a certain radius R of a city's center, we divide the circle into N thin concentric bands of thickness Δr. The band whose outer boundary is at a distance r_i from the center has area approximately $2\pi r_i \Delta r$, so the population residing within this band is $P_i \approx 2\pi r_i \rho(r_i)\Delta r$. Thus the total population is

$$P = \lim_{N\to\infty} \sum_{i=1}^{N} P_i = \lim_{N\to\infty} \sum_{i=1}^{N} 2\pi\rho(r_i)\,\Delta r = 2\pi \int_0^R r\rho(r)\,dr.$$

Example 4. The radial density function for a city's population (in thousands of people per square mile) is $\rho(r) = 20(1+r^2)^{-1/2}$. How many people live within 5 miles of the city's center?

$$P = \int_0^5 2\pi r \cdot 20(1+r^2)^{-1/2}\,dr$$

$u = 1 + r^2$

$du = 2r\,dr$

$$= 20\pi \int_1^{26} u^{-1/2}\,du$$

$$= 40\pi\, u^{1/2}\Big|_1^{26}$$

$$= 40\pi(\sqrt{26} - 1) \approx 515.098$$

So about 515,098 people live within 5 miles of the city's center

Average Value. Suppose we want to know the "average" or "typical" y-value of a function $f(x)$ on the interval $[a, b]$. Since there are infinitely many points to consider, we can't compute average in the usual way, but we can approximate by subdividing the interval and averaging the values at the right-hand endpoints:

$$f_{ave} \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{N} f(x_i)\Delta x,$$

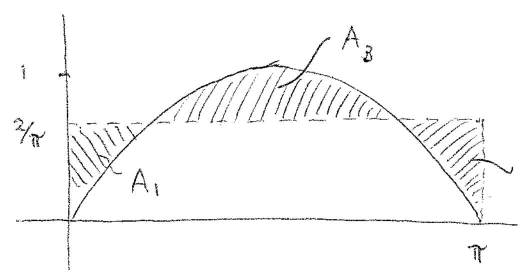
where $\Delta x = (b-a)/N$. Letting $N \to \infty$ gives

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For positive functions, f_{ave} is the height of a rectangle having the same area as the area under the curve.

Example 5. Find the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

$$\begin{aligned} f_{ave} &= \frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx \\ &= \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} \\ &= \frac{1}{\pi}(1+1) \\ &= \frac{2}{\pi} \end{aligned}$$



$A_1 + A_2 = A_3$

Recall that if $s(t)$ denotes the position of a particle at time t and $v(t)$ denotes the particle's instantaneous velocity, then by using the Fundamental Theorem of Calculus we find that the particle's **average velocity** on the interval $[a, b]$ is

$$\frac{s(b) - s(a)}{b - a} = \frac{1}{b-a}\int_a^b v(t)\,dt.$$

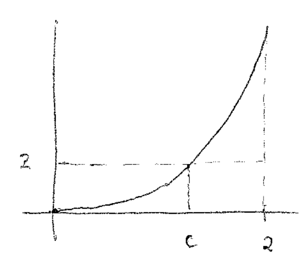
Hence our new definition of average value for a velocity function agrees with our old definition of average velocity.

Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that $f_{ave} = f(c)$.

Example 6. Find a a number c in the interval $[0, 2]$ such that the average value of $f(x) = x^3$ on $[0, 2]$ is equal to $f(c)$.

$$\begin{aligned} f_{ave} &= \frac{1}{2-0}\int_0^2 x^3\,dx \\ &= \frac{1}{2}\cdot\frac{x^4}{4}\Big|_0^2 \\ &= 2 \end{aligned}$$

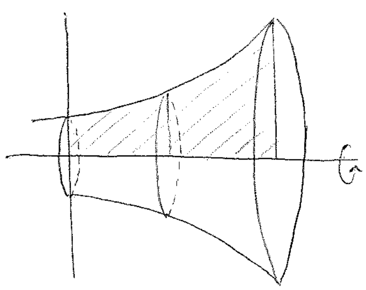
$f(c) = c^3 = 2 \Rightarrow c = \sqrt[3]{2} \approx 1.26$



§6.3—Volumes of Revolution

Given a region in the xy-plane, we can generate a 3-dimensional solid by rotating each point in the region about a given axis. For example, a sphere can be obtained by rotating the top half of a disk about the x-axis.

Example 1. Find the volume of the solid obtained by revolving the region bounded by the curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = 2$ about the x-axis.



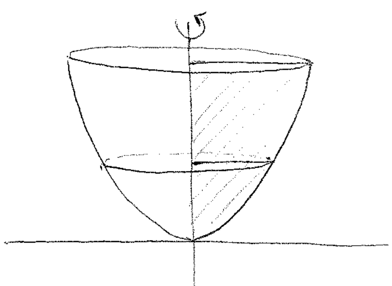
Cross-section at x is a disk of radius $y = e^x$

and area $A(x) = \pi (e^x)^2 = \pi e^{2x}$

$$V = \int_0^2 \pi e^{2x} \, dx$$

$$= \frac{\pi}{2} e^{2x} \Big|_0^2 = \frac{\pi}{2}(e^4 - 1)$$

Example 2. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the curve $y = x^2$ and the line $y = 4$ about the y-axis.



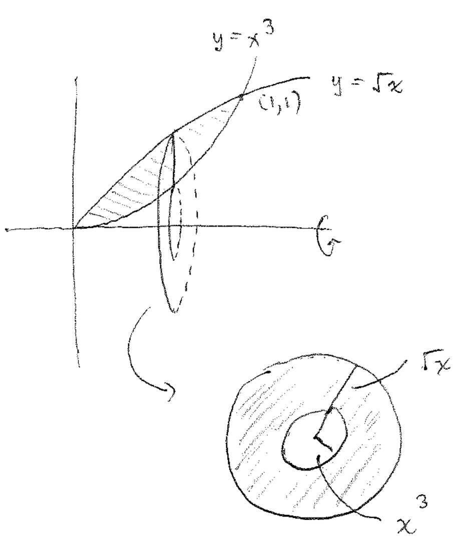
Cross-section at y is a disk of radius $x = \sqrt{y}$

and area $A(y) = \pi (\sqrt{y})^2 = \pi y$

$$V = \int_0^4 \pi y \, dy$$

$$= \pi \frac{y^2}{2} \Big|_0^4 = 8\pi$$

Example 3. Find the volume of the solid generated by revolving the region bounded by the curves $y = x^3$ and $y = \sqrt{x}$ about the x-axis.



Cross-section at x is a washer of area

$$A(x) = \pi(\sqrt{x})^2 - \pi(x^3)^2$$

$$= \pi(x - x^6)$$

$$V = \int_0^1 \pi(x - x^6) \, dx$$

$$= \pi\left(\frac{x^2}{2} - \frac{x^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{7}\right)$$

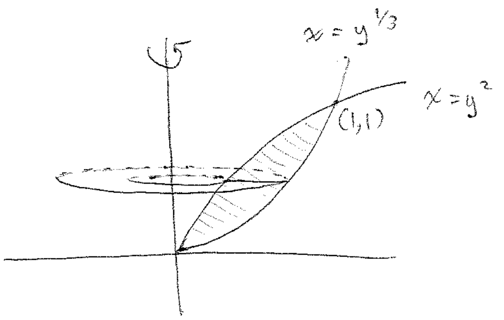
$$= \frac{5\pi}{14}$$

Basic formula for the disk/washer method about a horizontal axis:

$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\, dx,$$

where $R(x)$ is the outer radius and $r(x)$ is the inner radius. For a vertical axis, the integration will be with respect to y instead of x.

Example 4. Find the volume of the solid generated by revolving the region used in Example 3 about the y-axis.



Cross-section at y is a washer with outer radius $R(y) = y^{1/3}$ and inner radius $r(y) = y^2$

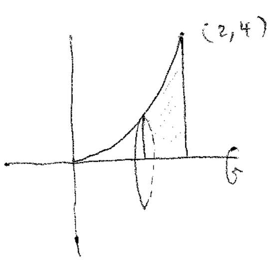
$$V = \int_0^1 \pi\left([y^{1/3}]^2 - [y^2]^2\right) dy$$

$$= \pi \int_0^1 (y^{2/3} - y^4)\, dy$$

$$= \pi\left(\frac{3}{5} y^{5/3} - \frac{1}{5} y^5\right)\Big|_0^1 = \frac{2\pi}{5}$$

Example 5. Let $\mathcal{R}$ be the region bounded by the curve $y = x^2$ and the lines $y = 0$ and $x = 2$. Set up integrals for the volumes of the solids generated by revolving $\mathcal{R}$ about

(a) the x-axis

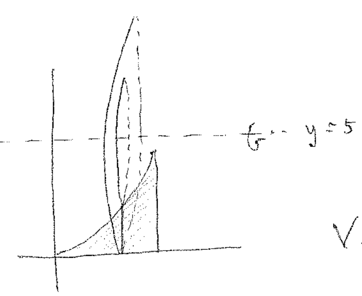


$R(x) = x^2$

$r(x) = 0$

$$V = \int_0^2 \pi x^4\, dx$$

(c) the line $y = 5$

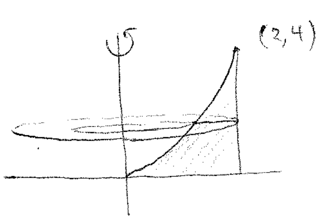


$R(x) = 5$

$r(x) = 5 - x^2$

$$V = \int_0^2 \pi\left(25 - (5 - x^2)^2\right) dx$$

(b) the y-axis

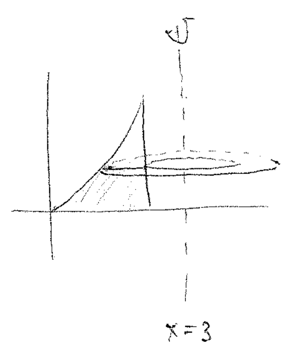


$R(y) = 2$

$r(y) = \sqrt{y}$

$$V = \int_0^4 \pi(4 - y)\, dy$$

(d) the line $x = 3$



$R(y) = 3 - \sqrt{y}$

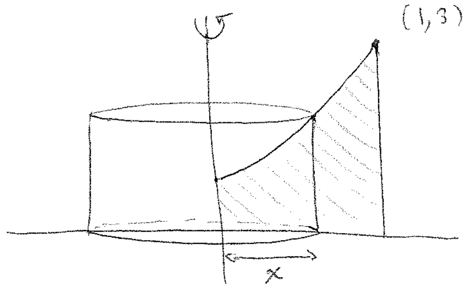
$r(y) = 1$

$$V = \int_0^4 \pi\left((3 - \sqrt{y})^2 - 1\right) dy$$

§6.4—The Method of Cylindrical Shells

In §6.3, we looked at slices *perpendicular* to the axis of revolution, and each slice traced out a disk or washer as it revolved about the axis. An alternative approach for computing volumes of revolution is to use slices *parallel* to the axis, which trace out cylindrical shells.

Example 1. Find the volume of the solid generated by revolving the region bounded by the curve $y = x^3 + x + 1$ and the lines $y = 0$, $x = 0$, and $x = 1$ about the y-axis.



Vertical strip at x traces out a cylindrical shell of radius x, height $y = x^3 + x + 1$, and thickness dx.

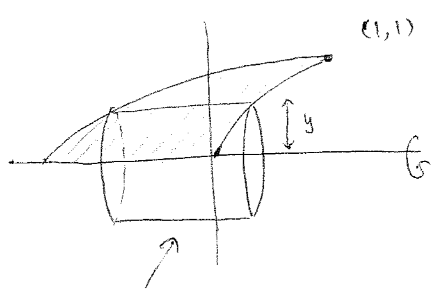
Volume of shell $\approx 2\pi x\,(x^3 + x + 1)\,dx$

$$V = \int_0^1 2\pi x\,(x^3 + x + 1)\,dx$$

$$= 2\pi \int_0^1 (x^4 + x^2 + x)\,dx = 2\pi\left(\frac{x^5}{5} + \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{5} + \frac{1}{3} + \frac{1}{2}\right) = \frac{31\pi}{15}$$

Example 2. Find the volume of the solid generated by revolving the region in the first and second quadrants bounded by the parabolas $x = 3y^2 - 2$ and $x = y^2$ about the x-axis.



$y^2 - (3y^2 - 2)$

$= 2 - 2y^2$

Shell at y has radius y, length $2 - 2y^2$, thickness dy, and volume $\approx 2\pi y\,(2 - 2y^2)\,dy$.

$$V = \int_0^1 2\pi y\,(2 - 2y^2)\,dy$$

$$= 2\pi \int_0^1 (2y - 2y^3)\,dy$$

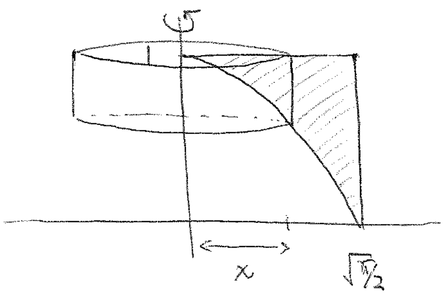
$$= 2\pi\left(y^2 - \frac{1}{2}y^4\right)\Big|_0^1 = 2\pi\left(1 - \frac{1}{2}\right) = \pi$$

Basic formula for the cylindrical shell method about a vertical axis:

$$V = \int_a^b 2\pi r(x)h(x)\,dx,$$

where $r(x)$ is the shell radius and $h(x)$ is the shell height. For a horizontal axis, the integration will be with respect to y instead of x.

Example 3. Find the volume of the solid obtained by revolving the region in the first quadrant bounded by the curve $y = \cos(x^2)$ and the lines $y = 1$, $x = 0$, and $x = \sqrt{\pi/2}$ about the y-axis.



Shell radius : $r(x) = x$

Shell height : $h(x) = 1 - \cos(x^2)$

$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \left(1 - \cos(x^2)\right) dx \qquad u = x^2, \quad du = 2x\,dx$$

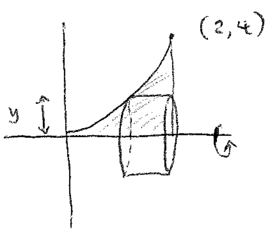
$$= \pi \int_0^{\pi/2} (1 - \cos u)\, du$$

$$= \pi (u - \sin u) \Big|_0^{\pi/2}$$

$$= \pi \left(\frac{\pi}{2} - 1\right)$$

Example 4. Let $\mathcal{R}$ be the region bounded by the curve $y = x^2$ and the lines $y = 0$ and $x = 2$, as considered in Example 5 from the §6.3 notes. Use the shell method to set up integrals for the volumes of the solids generated by revolving $\mathcal{R}$ about

(a) the x-axis

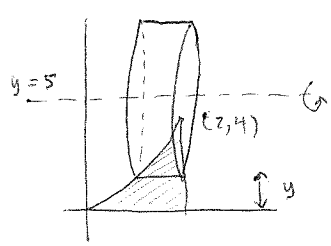


$r(y) = y$

$h(y) = 2 - \sqrt{y}$

$$V = \int_0^4 2\pi y \left(2 - \sqrt{y}\right) dy$$

(c) the line $y = 5$

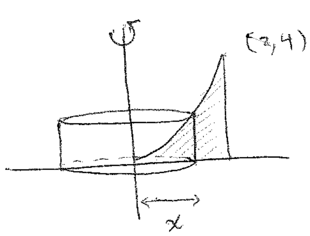


$r(y) = 5 - y$

$h(y) = 2 - \sqrt{y}$

$$V = \int_0^4 2\pi (5 - y)\left(2 - \sqrt{y}\right) dy$$

(b) the y-axis

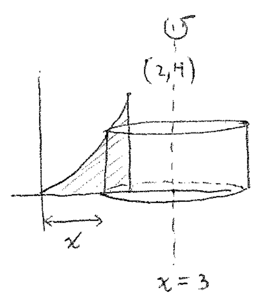


$r(x) = x$

$h(x) = x^2$

$$V = \int_0^2 2\pi x^3\, dx$$

(d) the line $x = 3$



$r(x) = 3 - x$

$h(x) = x^2$

$$V = \int_0^2 2\pi (3 - x)\, x^2\, dx$$

Washers versus Shells

Method	Axis of revolution	Types of slices	Integrate with respect to
Washers	horizontal line	vertical strips	x
Washers	vertical line	horizontal strips	y
Shells	horizontal line	horizontal strips	y
Shells	vertical line	vertical strips	x

Note that in the washer method the slices are always perpendicular to the axis of revolution and in the shell method the slices are always parallel to the axis of revolution.

§6.5—Work and Energy

Work. Recall that work is defined by the basic equation

$$\text{Work} = \text{Force} \times \text{Distance}$$

when the force is constant and acts in the direction of motion. The typical units for work are therefore Newton-meters (Joules) or foot-pounds. Work can be viewed as the energy expended in accomplishing a task.

If the force is not constant, then we need a slicing approach to calculate work. If $F(x)$ represents the force at position x, then the work done over the interval $[x, x + \Delta x]$ is approximately $F(x)\Delta x$, so the total work done over the interval $[a, b]$ is

$$W = \int_a^b F(x)\,dx.$$

Note that there is a strong analogy with the calculation of distance traveled from velocity. If velocity is constant, then distance = velocity × time, but if the velocity varies, then we need the integral of velocity to calculate distance.

Springs. A simple example of a varying force occurs when stretching a spring. Hooke's Law states that the force required to stretch or compress a string x units from its natural length is $F(x) = kx$, where k is a number known as the spring constant.

Example 1. A certain spring requires a force of 80 Newtons to stretch from its natural length of 2 meters to a length of 4 meters. How much work is required to stretch it from rest to a length of 5 meters?

Hooke's Law : $F(x) = kx \Rightarrow 80\text{ N} = k\,(4-2)\text{ m}$

$\Rightarrow k = 40\text{ N/m}$

So $F(x) = 40x$

Thus Work $= \int_0^3 40x\,dx$

$= 20x^2 \Big|_0^3 = 180\text{ J}$

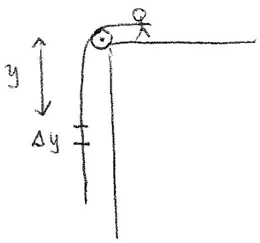
Vertical Ropes and Cables. Suppose that a cable or rope is used to raise a heavy object like a cement block from the ground to some given height. The work in raising the object can be calculated using Work = Force × Distance, but finding the work done in raising the cable is trickier. There are two ways to think about breaking the problem down:

(1) Consider raising the entire cable a small distance, OR

(2) Consider raising a small slice of the cable all the way to the top.

The first approach is what we used for springs, but the second approach is more useful for many of the applications we want to consider. Therefore we take the second approach in the next example.

Example 2. A uniform cable 20 feet long hangs from the top of a tall building. If the cable weighs 3 pounds per foot, find the work done in raising the entire cable to the top.



A piece of cable of length Δy at y feet from the top weighs $(3\ \text{lb/ft})(\Delta y\ \text{ft}) = 3\,\Delta y\ \text{lb}$ and moves a total distance of y feet.

Hence the work done on this little slice is

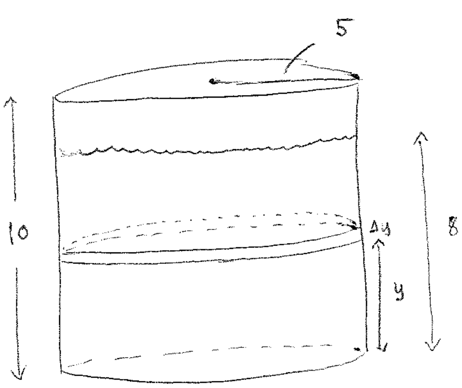
$$(3\,\Delta y\ \text{lb})\cdot(y\ \text{ft}) = 3\,y\,\Delta y\ \text{ft}\cdot\text{lb}.$$

The total work in raising the cable is therefore

$$W = \int_0^{20} 3y\,dy = \frac{3}{2}y^2\Big|_0^{20} = 600\ \text{ft}\cdot\text{lb}.$$

Pumping liquids from tanks. The method used in Example 2 (slicing the object into small pieces and moving each piece all the way to the top) applies very nicely to situations in which liquid is being pumped from a tank. The work integral that arises will depend on the geometry of the slices that occur in each particular problem.

Example 3. A cylindrical tank of height 10 meters and radius 5 meters is filled to a height of 8 meters with water, which weighs 9800 Newtons per cubic meter. Find the work required to pump all the water to the top of the tank.



Consider a slice of thickness Δy at height y.

Volume of slice $= \pi\cdot 5^2\cdot\Delta y = 25\pi\,\Delta y\ \text{m}^3$

Weight of slice (force) $= 9800\ \text{N/m}^3 \cdot 25\pi\,\Delta y\ \text{m}^3 = 9800\cdot 25\pi\,\Delta y\ \text{N}$

Distance moved by slice $= 10-y\ \text{m}$

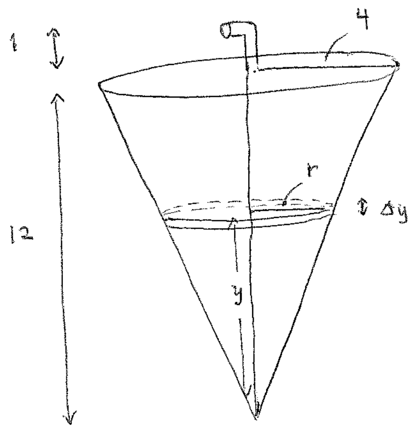
Work on slice $= 9800\cdot 25\pi\,(10-y)\,\Delta y\ \text{N}\cdot\text{m}$

$$\text{Total Work} = \int_0^8 9800\cdot 25\pi\,(10-y)\,dy$$

$$= 9800\cdot 25\pi\left(10y - \frac{y^2}{2}\right)\Big|_0^8$$

$$= 9800\cdot 25\pi\,(80-32) \approx 36{,}945{,}130\ \text{J}$$

Example 4. A tank has the shape of a cone (with the vertex at bottom) with height 12 feet and radius 4 feet. If the tank is completely filled with water, which weighs 62.4 pounds per cubic foot, find the work required to pump all the water out through a spout that extends 1 foot above the tank's top.



Volume of slice $= \pi r^2 \Delta y = \pi \left(\frac{1}{3}y\right)^2 \Delta y$

Weight of slice $= 62.4 \cdot \frac{\pi}{9} y^2 \Delta y$

Distance moved by slice $= 13 - y$

Work on slice $= \frac{62.4\pi}{9} y^2 (13-y)\, \Delta y$

$$\text{Total Work} = \int_0^{12} \frac{62.4\pi}{9} y^2 (13-y)\, dy$$

$$= \frac{62.4\pi}{9} \int_0^{12} (13y^2 - y^3)\, dy$$

$$= \frac{62.4\pi}{9} \left(\frac{13}{3} y^3 - \frac{1}{4} y^4 \right) \Big|_0^{12}$$

$$\approx 50{,}185 \text{ ft-lb}$$

similar Δ's

$\Rightarrow \quad \frac{r}{y} = \frac{4}{12}$

$\Rightarrow \quad r = \frac{1}{3} y$

§8.2—Fluid Pressure and Force

Hydrostatic Pressure and Force. The pressure (or force per unit area) of a fluid at a given depth below the surface is determined by the equation

$$\text{Pressure} = \text{Weight Density} \times \text{Depth}.$$

The weight density typically has units of N/m^3 or lb/ft^3, so that the units of pressure are N/m^2 (Pascals) or lb/ft^2. We'll often find it useful to recall that the weight density of water is approximately

$$62.4 \text{ lb/ft}^3 \qquad \text{or} \qquad 9800 \text{ N/m}^3.$$

The force on a thin horizontal plate submerged in a liquid can be calculated using the equation

$$\text{Force} = \text{Pressure} \times \text{Area},$$

since the pressure is constant along the plate. At a fixed depth, the pressure exerted by a fluid is the same in all directions, so a force of this magnitude is exerted on both the top and the bottom of the plate. If the plate is submerged vertically, then the pressure is not constant throughout, so the force on it must be calculated by slicing, and this leads to an integral.

Example 1. Consider a swimming pool 5 meters wide, 10 meters long, and 3 meters deep. If the pool is completely filled with water, find the force on

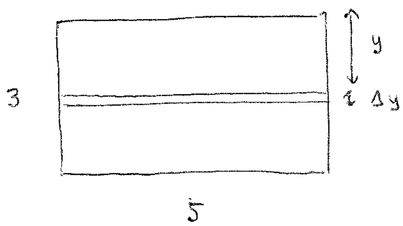
(a) the bottom of the pool.

$$\text{Pressure along bottom} = (9800\ \text{N}/\text{m}^3)\cdot(3\ \text{m}) = 29{,}400\ \text{N}/\text{m}^2$$

$$\text{Area of bottom} = 50\ \text{m}^2$$

$$\text{Force on bottom} = (29{,}400\ \text{N}/\text{m}^2)(50\ \text{m}^2) = 1{,}470{,}000\ \text{N}$$

(b) one of the 5×3 ends of the pool.



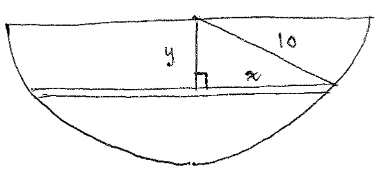
$$\text{Pressure along strip} = 9800\,y$$

$$\text{Area of strip} = 5\,\Delta y$$

$$\text{Force on strip} = 49{,}000\,y\,\Delta y$$

$$\text{Total Force} = \int_0^3 49{,}000\,y\,dy = 49{,}000\cdot\frac{y^2}{2}\Big|_0^3 = 220{,}500\ \text{N}$$

Example 2. The vertical ends of a tank are semi-circles of radius 10 feet with diameter running along the top. If the tank is filled to the top with water, find the force on one end of the tank.



Pythagorean Theorem

$$\Rightarrow\quad x^2+y^2 = 100$$

$$\Rightarrow\quad x = \sqrt{100-y^2}$$

$$\text{Pressure along strip} = 62.4\,y$$

$$\text{Area of strip} = 2x\,\Delta y = 2\sqrt{100-y^2}\,\Delta y$$

$$\text{Force on strip} = 124.8\,y\sqrt{100-y^2}\,\Delta y$$

$$\text{Total Force} = \int_0^{10} 124.8\,y\sqrt{100-y^2}\,dy$$

$u = 100-y^2$

$du = -2y\,dy$

$$= -62.4\int_{100}^{0}\sqrt{u}\,du$$

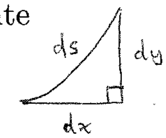
$$= -62.4\cdot\frac{2}{3}u^{3/2}\Big|_{100}^{0}$$

$$= 62.4\cdot\frac{2}{3}\cdot 1000$$

$$= 41{,}600\ \text{lb}$$

§8.1—Arc Length and Surface Area

Suppose we want to compute the length of a curve in the plane. If we think of ds as the length of a small piece of the curve, then we can use the Pythagorean Theorem to write



$$ds \approx \sqrt{(dx)^2 + (dy)^2}.$$

If the curve is given by the function $y = f(x)$ on $[a, b]$ then we have $dy = f'(x)dx$, so

$$ds \approx \sqrt{(dx)^2 + (f'(x)dx)^2} = \sqrt{(dx)^2(1 + [f'(x)]^2)} = \sqrt{1 + [f'(x)]^2}\, dx$$

The total length of the curve is then given by

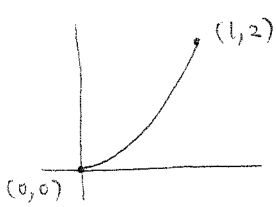
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Once again, we view the integral as adding up the contributions from the individual pieces.

Example 1. Find the length of the curve $y = 2x^{3/2}$ from $x = 0$ to $x = 1$.

$f(x) = 2x^{3/2}$

$f'(x) = 3x^{1/2}$



$$L = \int_0^1 \sqrt{1 + (3x^{1/2})^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 9x}\, dx \qquad u = 1 + 9x, \quad du = 9\, dx$$

$$= \frac{1}{9} \int_1^{10} \sqrt{u}\, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

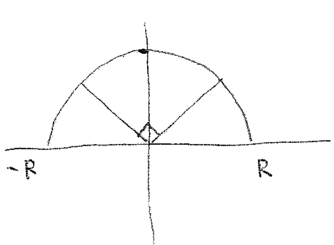
$$= \frac{2}{27}\left(10^{3/2} - 1\right) \approx 2.268$$

Example 2. Find the circumference of a circle of radius R.

$f(x) = \sqrt{R^2 - x^2}$

$f'(x) = \frac{1}{2}(R^2 - x^2)^{-1/2}(-2x)$

$= \dfrac{-x}{\sqrt{R^2 - x^2}}$



$$C = 4 \int_{-R/\sqrt{2}}^{R/\sqrt{2}} \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx$$

$$= 8 \int_0^{R/\sqrt{2}} \sqrt{\frac{R^2}{R^2 - x^2}}\, dx$$

$$= 8R \int_0^{R/\sqrt{2}} \frac{1}{\sqrt{R^2 - x^2}}\, dx$$

$$= 8R \sin^{-1}\left(\frac{x}{R}\right) \Big|_0^{R/\sqrt{2}}$$

$$= 8R \left(\sin^{-1}(1/\sqrt{2}) - \sin^{-1}(0)\right)$$

$$= 8R\left(\pi/4 - 0\right)$$

$$= 2\pi R$$

If we revolve a curve $y = f(x)$ on the interval $[a, b]$ about the x-axis, then we can calculate the area of the resulting surface by breaking the curve into pieces, just as we did for arc length. A piece of the curve of length ds at an average distance y from the x-axis traces out a surface that is well-approximated by a slice of a cone whose area is approximately $2\pi y\, ds$. Thus we find that the area of the entire surface of revolution is

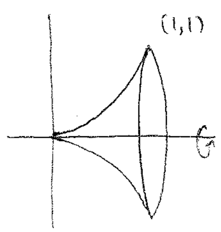
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

Similar formulas can be obtained for rotation about other axes.

Example 3. Find the area of the surface obtained by revolving the portion of the curve $y = x^3$ from $x = 0$ to $x = 1$ about the x-axis.

$f(x) = x^3$

$f'(x) = 3x^2$



$$S = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$$= \frac{\pi}{18} \int_1^{10} \sqrt{u}\, du$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

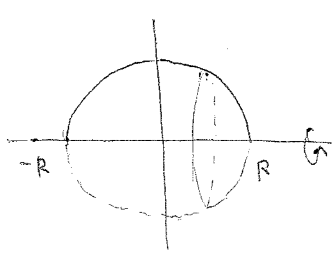
$$= \frac{\pi}{27} \left(10^{3/2} - 1\right) \approx 3.563$$

Example 4. Find the surface area of a sphere of radius R.

$f(x) = \sqrt{R^2 - x^2}$

$f'(x) = \frac{1}{2}(R^2 - x^2)^{-1/2}(-2x)$

$= \dfrac{-x}{\sqrt{R^2 - x^2}}$



$$S = 2 \int_0^R 2\pi \sqrt{R^2 - x^2} \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx$$

$$= 4\pi \int_0^R \sqrt{R^2 - x^2 + x^2}\, dx$$

$$= 4\pi \int_0^R \sqrt{R^2}\, dx$$

$$= 4\pi R \int_0^R dx$$

$$= 4\pi R^2$$

§7.1—Numerical Integration

Many functions (for example, e^{x^2} and $\sqrt{1+x^3}$) do not have elementary antiderivatives. The Fundamental Theorem of Calculus does not allow us to evaluate definite integrals of such functions exactly, so we must instead rely on approximations. For purposes of illustration, we will work with an integral that we *can* evaluate exactly:

$$\int_0^1 \frac{4}{1+x^2}\,dx = 4\tan^{-1} x\Big|_0^1 = \pi \approx 3.14159265358.$$

This will allow us to get a feel for the relative accuracy of our various approximation methods. The methods are all based on the fact that the definite integral is a limit of Riemann sums. When N subdivisions are used to approximate $\int_a^b f(x)\,dx$, we write $\Delta x = (b-a)/N$ for the width of each subinterval and $x_j = a + j\Delta x$ for the right-hand endpoint of the jth subinterval.

Left and Right Sums. We have already discussed approximations based on rectangles whose heights were determined by the value of the function at the left or right endpoint of each subinterval:

$$L_N = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{N-1})] \quad \text{and} \quad R_N = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_N)].$$

Example 1. Compute the approximations L_{10} and R_{10} for the integral $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$.

$L_{10} = 0.1\,[\,f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)\,]$

≈ 3.239926

$R_{10} = 0.1\,[\,f(0.1) + f(0.2) + f(0.3) + \cdots + f(1)\,]$

≈ 3.039926

The Midpoint and Trapezoidal Rules. We often get a more accurate estimate by choosing the rectangle heights to be the value of f at the midpoints

$$\bar{x}_j = \frac{1}{2}(x_{j-1} + x_j)$$

of the subintervals. This gives the approximation

$$M_N = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_N)].$$

A similar idea is to average the left and right-hand approximations; this is equivalent to approximating with trapezoids and gives the estimate

$$T_N = \tfrac{1}{2}(L_N + R_N) = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)].$$

Example 2. Compute the approximations M_{10} and T_{10} for the integral $\int_0^1 \frac{4}{1+x^2}\,dx$.

$$M_{10} = 0.1\,[\,f(0.05) + f(0.15) + f(0.25) + \dots + f(0.95)\,]$$

$$\approx 3.142460$$

$$T_{10} = \frac{0.1}{2}\,[\,f(0) + 2f(0.1) + 2f(0.2) + \dots + 2f(0.9) + f(1)\,]$$

$$\approx 3.139926$$

Error Estimates. If $|f''(x)| \le K_2$ for all x in $[a,b]$, then the errors in the trapezoidal and midpoint rules satisfy

$$|E_T| \le \frac{K_2(b-a)^3}{12N^2} \quad \text{and} \quad |E_M| \le \frac{K_2(b-a)^3}{24N^2}.$$

Example 3. Consider the integral $\int_2^5 \ln x\,dx$.

(a) Compute upper bounds for the error in using the trapezoidal and midpoint rules with $N = 6$ to approximate the integral.

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$|f''(x)| = \frac{1}{x^2} \le \frac{1}{2^2} = \frac{1}{4}$ on $[2,5]$

So can take $K_2 = \frac{1}{4}$

$$|E_T| \le \frac{\frac{1}{4}(5-2)^3}{12\cdot 6^2} = 0.015625$$

$$|E_M| \le \frac{\frac{1}{4}(5-2)^3}{24\cdot 6^2} = 0.0078125$$

(b) How large must N be in order to guarantee that $|E_T| \le 10^{-5}$?

Need $\frac{\frac{1}{4}(5-2)^3}{12N^2} \le 10^{-5}$

$$\Leftrightarrow \quad N^2 \ge \frac{10^5\cdot 27}{48} = 56250$$

$$\Leftrightarrow \quad N > 237$$

Simpson's Rule. Because the midpoint rule tends to be about twice as accurate as the trapezoidal rule and the errors tend to have opposite signs, we can expect to improve our approximations by taking a weighted average. When N is even, the Simpson's Rule approximation is given by

$$S_N = \tfrac{1}{3}T_{N/2}+\tfrac{2}{3}M_{N/2} = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{N-2})+4f(x_{N-1})+f(x_N)].$$

This method is equivalent to approximating f by parabolas on each subinterval.

Example 4. Compute S_{10} for the integral $\int_0^1 \frac{4}{1+x^2}\,dx$.

$$S_{10} = \frac{0.1}{3}\left[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)\right]$$

$$\approx 3.141592614$$

Example 5. Compute S_6 for the integral $\int_0^2 e^{-x^2}\,dx$. $\Delta x = 1/3$

$$S_6 = \frac{1/3}{3}\left[f(0) + 4f(1/3) + 2f(2/3) + 4f(1) + 2f(4/3) + 4f(5/3) + f(2)\right]$$

$$\approx 0.88203$$

The Error in Simpson's Rule. If $|f^{(4)}(x)| \le K_4$ for all x in $[a,b]$ then the error in Simpson's Rule satisfies

$$|E_S| \le \frac{K_4(b-a)^5}{180N^4}.$$

Example 6. Estimate the error in using S_6 to approximate $\int_2^5 \ln x\,dx$, and determine how large N must be in order to guarantee that $|E_S| \le 10^{-5}$.

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = \frac{-6}{x^4}$

$|f^{(4)}(x)| = \frac{6}{x^4} \le \frac{6}{2^4} = \frac{3}{8}$ on $[2,5]$

so can take $K_4 = 3/8$

$$|E_S| \le \frac{\frac{3}{8}(5-2)^5}{180\cdot 6^4} \approx 0.00039$$

To get $\frac{\frac{3}{8}\cdot 3^5}{180N^4} \le 10^{-5}$ need $N^4 \ge \frac{10^5\cdot 3^6}{8\cdot 180}$

$\iff N \ge 15$

§7.2—Integration by Parts

It is not true that the integral of a product is the product of the integrals. To deal with integrals of products, we can try to reverse the product rule:

$$(uv)' = uv' + vu' \implies uv = \int uv' + \int vu' \implies \int uv' = uv - \int vu'.$$

We usually write this in the form

$$\int u\,dv = uv - \int v\,du.$$

The point is that we can integrate the product $u\,dv$ if we know how to integrate $v\,du$.

Example 1. Evaluate $\displaystyle\int xe^x\,dx$.

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

Example 2. Evaluate $\displaystyle\int x\sin 3x\,dx$.

$u = x \qquad dv = \sin 3x\,dx$

$du = dx \qquad v = -\frac{1}{3}\cos 3x$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

Example 3. Evaluate $\displaystyle\int_1^e \ln x\,dx$.

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$$= x\ln x\Big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx$$

$$= e\ln e - 1\ln 1 - \int_1^e dx$$

$$= e - (e-1)$$

$$= 1$$

How to choose u and dv. When making this decision, we should keep in mind that we are essentially trading in the problem $\int u\,dv$ for the alternative problem $\int v\,du$. Thus we should consider the following:

1. We **must** know how to integrate dv; otherwise we cannot find v.
2. It is often helpful if v is simpler (or at least no more complicated) than dv.
3. It is often helpful if du is simpler (or at least no more complicated) than u.

Notice that Examples 1 and 2 are classic cases of this strategy, letting u be a power function and dv be an exponential, sine, or cosine. In Example 3, we followed Recommendation #3 but not #2 because what really matters (and what we were able to achieve) is that the **product** $v\,du$ is simpler than the original product $u\,dv$.

Example 4. Evaluate $\displaystyle\int x^2 \cos 5x\,dx$.

① $u = x^2$ $\quad dv = \cos 5x\,dx$

$du = 2x\,dx$ $\quad v = \frac{1}{5}\sin 5x$

② $u = x$ $\quad dv = \sin 5x\,dx$

$du = dx$ $\quad v = -\frac{1}{5}\cos 5x$

$$= \frac{1}{5}x^2 \sin 5x - \frac{2}{5}\int x \sin 5x\,dx$$

$$= \frac{1}{5}x^2 \sin 5x - \frac{2}{5}\left[-\frac{1}{5}x\cos 5x + \frac{1}{5}\int \cos 5x\,dx\right]$$

$$= \frac{1}{5}x^2 \sin 5x + \frac{2}{25}x\cos 5x - \frac{2}{25}\int \cos 5x\,dx$$

$$= \frac{1}{5}x^2 \sin 5x + \frac{2}{25}x\cos 5x - \frac{2}{125}\sin 5x + C$$

Example 5. Evaluate $\displaystyle\int \sin^{-1} x\,dx$.

$u = \sin^{-1}x$ $\quad dv = dx$

$du = \dfrac{dx}{\sqrt{1-x^2}}$ $\quad v = x$

$w = 1-x^2$

$dw = -2x\,dx$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\int \frac{dw}{\sqrt{w}}$$

$$= x\sin^{-1}x + \frac{1}{2}\cdot 2w^{1/2} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

Example 6. Evaluate $\int \cos(\sqrt{x})\,dx$ by first making the substitution $w = \sqrt{x}$.

$$dw = \frac{1}{2} x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow \quad dx = 2\sqrt{x}\,dw = 2w\,dw$$

$$\int \cos(\sqrt{x})\,dx = \int 2w \cos w\,dw$$

$u = 2w \qquad dv = \cos w\,dw$

$du = 2\,dw \qquad v = \sin w$

$$\int \cos(\sqrt{x})\,dx = 2w \sin w - \int 2 \sin w\,dw$$

$$= 2w \sin w + 2\cos w + C$$

$$= 2\sqrt{x} \sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$$

Example 7. Evaluate the integral $I = \int e^x \cos x\,dx$.

1

$u = e^x \qquad dv = \cos x\,dx$

$du = e^x\,dx \qquad v = \sin x$

2

$u = e^x \qquad dv = \sin x\,dx$

$du = e^x\,dx \qquad v = -\cos x$

$$I = e^x \sin x - \int e^x \sin x\,dx$$

$$= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\,dx \right]$$

$$= e^x \sin x + e^x \cos x - I$$

Boomerang!

Now we can use algebra to solve for I:

$$2I = e^x \sin x + e^x \cos x$$

$$\Rightarrow \quad I = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C$$

Note that the "+ C" was temporarily lost when we ignored the fact that the two "I"'s can differ by a constant.

§7.3—Trigonometric Integrals

By making use of basic trig identities in combination with the substitution method, we can handle many integrals involving powers of trig functions. The manipulations are all based on the Pythagorean identities

$$\cos^2 x + \sin^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad \cot^2 x + 1 = \csc^2 x$$

and the double-angle formulas

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{and} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}.$$

The strategies differ according to whether we have even or odd exponents–see the text for the general algorithm. The following examples are meant to illustrate the underlying ideas.

Example 1. Evaluate $\int \sin^2 x \cos^5 x\, dx$.

$$= \int \sin^2 x \, (\cos^2 x)^2 \cos x \, dx$$

$$= \int \sin^2 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^2 (1 - u^2)^2 \, du$$

$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{3} u^3 - \frac{2}{5} u^5 + \frac{1}{7} u^7 + C$$

$$= \frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + C$$

Example 2. Evaluate $\int \cos^4 x\, dx$.

$$= \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right) dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2} \cos 4x \right) dx$$

$$= \frac{1}{4} \left(\frac{3}{2} x + \sin 2x + \frac{1}{8} \sin 4x \right) + C$$

$$= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

Example 3. Evaluate $\int \tan^3 x \sec^4 x \, dx$.

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^3 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^3 (1 + u^2) \, du$$

$$= \int (u^3 + u^5) \, du$$

$$= \frac{1}{4} u^4 + \frac{1}{6} u^6 + C$$

$$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$$

§7.4—Trigonometric Substitution

The Pythagorean identities also allow us to convert certain algebraic expressions into powers of trig functions by making a trig substitution.

Example 1. Evaluate $\int \frac{dx}{x^2\sqrt{1-x^2}}$.

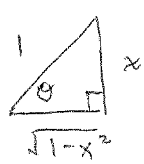
$x = \sin \theta$

$dx = \cos \theta \, d\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta}$

$= \cos \theta$



$$= \int \frac{\cos\theta \, d\theta}{\sin^2\theta \, \cos\theta}$$

$$= \int \frac{d\theta}{\sin^2\theta}$$

$$= \int \csc^2\theta \, d\theta$$

$$= -\cot\theta + C$$

$$= -\frac{\sqrt{1-x^2}}{x} + C$$

Example 2. Evaluate $\int \frac{dx}{x^2\sqrt{x^2-9}}$.

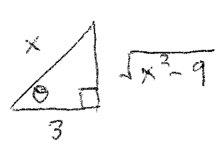
$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta$

$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9}$

$= \sqrt{9\tan^2\theta}$

$= 3\tan\theta$



$$= \int \frac{3\sec\theta\tan\theta \, d\theta}{9\sec^2\theta \cdot 3\tan\theta}$$

$$= \frac{1}{9}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{9}\int \cos\theta \, d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{1}{9} \cdot \frac{\sqrt{x^2-9}}{x} + C$$

Example 3. Evaluate $\displaystyle\int_0^2 \frac{x^3\,dx}{\sqrt{x^2+4}}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4}$

$= \sqrt{4\sec^2\theta}$

$= 2\sec\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= \int_0^{\pi/4} \frac{8\tan^3\theta \cdot 2\sec^2\theta\,d\theta}{2\sec\theta}$$

$$= \int_0^{\pi/4} 8\tan^3\theta\sec\theta\,d\theta$$

$$= 8\int_0^{\pi/4} (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$$= 8\int_1^{\sqrt{2}} (u^2-1)\,du$$

$$= 8\left(\tfrac{1}{3}u^3 - u\right)\Big|_1^{\sqrt{2}}$$

$$= 8\left(\tfrac{2}{3}\sqrt{2} - \sqrt{2} - \left(\tfrac{1}{3} - 1\right)\right)$$

$$= \tfrac{8}{3}\left(2 - \sqrt{2}\right)$$

The 3 basic substitutions.

1. Forms involving $\sqrt{a^2-x^2}$, where $-a \le x \le a$.
   Let $x = a\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$.
2. Forms involving $\sqrt{x^2-a^2}$, where $x \ge a$.
   Let $x = a\sec\theta$, where $0 \le \theta < \pi/2$. Then $\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta$.
3. Forms involving $\sqrt{a^2+x^2}$.
   Let $x = a\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $\sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2\theta)} = a\sec\theta$.

Example 4. Evaluate $\displaystyle\int \sqrt{16-x^2}\,dx$.

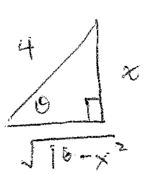
$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$\sqrt{16-x^2} = \sqrt{16 - 16\sin^2\theta}$

$= 4\sqrt{1-\sin^2\theta}$

$= 4\cos\theta$



$$= \int 4\cos\theta \cdot 4\cos\theta\,d\theta$$

$$= 16\int \cos^2\theta\,d\theta$$

$$= 8\int (1+\cos 2\theta)\,d\theta$$

$$= 8\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$

$$= 8\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= 8\left(\sin^{-1}\left(\tfrac{x}{4}\right) + \frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}\right) + C$$

$$= 8\sin^{-1}\left(\tfrac{x}{4}\right) + \tfrac{1}{2}x\sqrt{16-x^2} + C$$

Example 5. Evaluate $\displaystyle\int_{5\sqrt{2}}^{10} \frac{\sqrt{x^2-25}}{x}\,dx$.

$x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\sqrt{x^2-25} = \sqrt{25\sec^2\theta - 25} = 5\sqrt{\sec^2\theta - 1} = 5\tan\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{5\tan\theta \cdot 5\sec\theta\tan\theta\,d\theta}{5\sec\theta}$$

$$= 5\int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta$$

$$= 5\int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)\,d\theta$$

$$= 5(\tan\theta - \theta)\Big|_{\pi/4}^{\pi/3}$$

$$= 5\left(\sqrt{3} - \frac{\pi}{3} - \left(1 - \frac{\pi}{4}\right)\right) = 5\left(\sqrt{3} - 1 - \frac{\pi}{12}\right)$$

Example 6. Evaluate $\displaystyle\int \frac{dx}{(x^2+4x+7)^{5/2}}$.

Complete the square:

$$x^2+4x+7 = x^2+4x+4+3 = (x+2)^2+3$$

Let $u = x+2$

$du = dx$

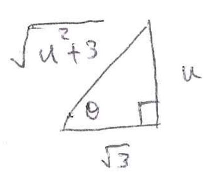
Now let $u = \sqrt{3}\tan\theta$

$du = \sqrt{3}\sec^2\theta\,d\theta$

$$(u^2+3)^{5/2} = (3\tan^2\theta + 3)^{5/2} = 3^{5/2}(\sec^2\theta)^{5/2} = 9\sqrt{3}\sec^5\theta$$

Finally, $w = \sin\theta$

$dw = \cos\theta\,d\theta$



$$= \int \frac{du}{(u^2+3)^{5/2}}$$

$$= \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sqrt{3}\sec^5\theta}$$

$$= \frac{1}{9}\int \frac{1}{\sec^3\theta}\,d\theta$$

$$= \frac{1}{9}\int \cos^3\theta\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\,(1-\sin^2\theta)\,d\theta$$

$$= \frac{1}{9}\int (1-w^2)\,dw$$

$$= \frac{1}{9}\left(w - \frac{1}{3}w^3\right) + C$$

$$= \frac{1}{9}\sin\theta - \frac{1}{27}\sin^3\theta + C$$

$$= \frac{1}{9}\cdot\frac{u}{\sqrt{u^2+3}} - \frac{1}{27}\cdot\frac{u^3}{(u^2+3)^{3/2}} + C$$

$$= \frac{x+2}{9\sqrt{x^2+4x+7}} - \frac{(x+2)^3}{27(x^2+4x+7)^{3/2}} + C$$

§7.6—The Method of Partial Fractions

We often simplify the sum or difference of rational functions by getting a common denominator. For example,

$$\frac{1}{x+2} - \frac{1}{x+5} = \frac{(x+5)-(x+2)}{(x+2)(x+5)} = \frac{3}{x^2+7x+10}.$$

If we want to integrate this function, however, the form on the left is actually simpler to deal with: Therefore we want to learn how to reverse the process of getting a common denominator; this is known as the method of partial fractions.

Example 1. Evaluate $\displaystyle\int \frac{1}{x^2+2x}\,dx$.

$$\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$\Rightarrow \quad 1 = A(x+2) + Bx$$

$x=0 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$x=-2 \Rightarrow -2B = 1 \Rightarrow B = -\frac{1}{2}$

So $\displaystyle\int \frac{1}{x^2+2x}\,dx = \frac{1}{2}\int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{x+2}$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

$$= \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$$

Example 2. Evaluate $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$.

$$\frac{2x+1}{x^2-7x+12} = \frac{2x+1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

$$\Rightarrow \quad 2x+1 = A(x-4) + B(x-3)$$

$x=3 \Rightarrow 7 = -A \Rightarrow A = -7$

$x=4 \Rightarrow 9 = B$

So $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx = -7\int\frac{dx}{x-3} + 9\int\frac{dx}{x-4}$

$$= -7\ln|x-3| + 9\ln|x-4| + C$$

$$= \ln\left(\frac{|x-4|^9}{|x-3|^7}\right) + C$$

If the denominator does not break down into distinct linear factors, we need to modify the form of the partial fraction expansion. The general procedure is as follows:

1. If the degree of the numerator is greater than or equal to the degree of the denominator, do long division.

2. Factor the denominator completely.

3. For each factor $(x-r)^m$, allow the terms $\dfrac{A_1}{x-r}+\dfrac{A_2}{(x-r)^2}+\cdots+\dfrac{A_m}{(x-r)^m}$.

4. For each factor $(x^2+px+q)^n$, where x^2+px+q is irreducible, allow the terms

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}.$$

5. Clear denominators, solve for the constants and integrate.

Example 3. Write down the form of the partial fraction decomposition for

$$R(x)=\frac{x^5+4x+1}{(x-2)(x+3)^3(x^2+5)(x^2+x+1)^2}.$$

$$R(x) = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)^3} + \frac{Ex+F}{x^2+5}$$

$$+ \frac{Gx+H}{x^2+x+1} + \frac{Ix+J}{(x^2+x+1)^2}$$

Example 4. Evaluate $\displaystyle\int \frac{x^2+1}{x^3+4x^2+4x}\,dx.$

$$\frac{x^2+1}{x^3+4x^2+4x} = \frac{x^2+1}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Rightarrow \quad x^2+1 = A(x+2)^2 + Bx(x+2) + Cx$$

$$x=0 \Rightarrow 1 = A(0+2)^2 \Rightarrow A = \frac{1}{4}$$

$$x=-2 \Rightarrow 5 = C(-2) \Rightarrow C = -\frac{5}{2}$$

$$x=1 \Rightarrow 2 = 9A+3B+C$$

$$\Rightarrow B = \frac{1}{3}(2-9A-C) = \frac{3}{4}$$

So $\displaystyle\int \frac{x^2+1}{x^3+4x^2+4x}\,dx$

$$= \frac{1}{4}\int \frac{1}{x}\,dx + \frac{3}{4}\int\frac{1}{x+2}\,dx - \frac{5}{2}\int\frac{1}{(x+2)^2}\,dx$$

$$= \frac{1}{4}\ln|x| + \frac{3}{4}\ln|x+2| + \frac{5}{2(x+2)} + C$$

Note: $\displaystyle\int\frac{1}{(x+2)^2}\,dx = \int u^{-2}\,du = \frac{u^{-1}}{-1}+C = -\frac{1}{x+2}+C$

$u = x+2$
$du = dx$

Example 5. Evaluate $\displaystyle\int \frac{x^3-2}{x^3+4x}\,dx$.

Long division:

$$\begin{array}{r} 1 \\ x^3+4x \,\big)\, x^3 - 2 \\ -(x^3+4x) \\ \hline -4x-2 \end{array}$$

So

$$\int \frac{x^3-2}{x^3+4x}\,dx = \int 1\,dx - \int \frac{4x+2}{x^3+4x}\,dx$$

Now

$$\frac{4x+2}{x^3+4x} = \frac{4x+2}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow \quad 4x+2 = A(x^2+4) + (Bx+C)x$$

equate like coeffs:

$$\Rightarrow \quad \underline{0}x^2 + \underline{4}x + \underline{2} = \underline{(A+B)}x^2 + \underline{C}x + \underline{4A}$$

$$\Rightarrow \begin{cases} A+B = 0 \\ C = 4 \\ 4A = 2 \end{cases} \quad \Rightarrow \quad A = 1/2, \quad B = -1/2, \quad C = 4$$

Thus

$$\int \frac{x^3-2}{x^3+4x}\,dx = \int 1\,dx - \left[\frac{1}{2}\int \frac{1}{x}\,dx + \int\frac{-\frac{1}{2}x+4}{x^2+4}\,dx\right]$$

$$= x - \frac{1}{2}\ln|x| + \frac{1}{2}\int \frac{x}{x^2+4}\,dx - 4\int \frac{dx}{x^2+4}$$

$u = x^2+4$

$du = 2x\,dx$

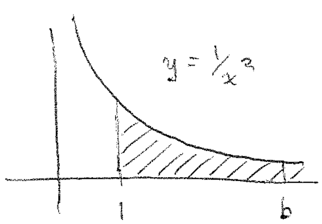
$$= x - \frac{1}{2}\ln|x| + \frac{1}{4}\ln(x^2+4) - 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

§7.7—Improper Integrals

We can make sense of integrals in which one limit is infinite by evaluating the integral up to some finite number b and then taking the limit of our answer as $b \to \infty$.

Example 1. Evaluate each of the following.

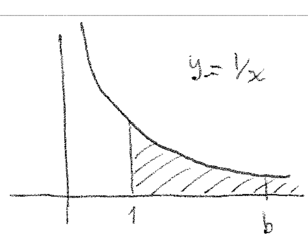
(a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$



$$= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \qquad \underline{\text{converges}}$$

(b) $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$



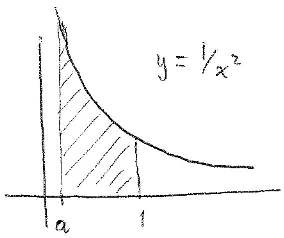
$$= \lim_{b\to\infty} \ln|x|\Big|_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1) = \infty \qquad \underline{\text{diverges}}$$

A similar approach can be applied to handle integrals in which the integrand has a vertical asymptote at one endpoint. For example, to integrate $1/x^2$ from 0 to 1 we can integrate from a to 1 (where a is a small positive number) and then take the limit as $a \to 0^+$.

Example 2. Evaluate each of the following.

(a) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx \;=\; \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx$

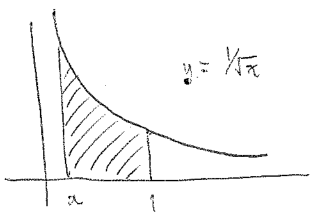


$$= \lim_{a\to 0^+} -\frac{1}{x}\Big|_a^1$$

$$= \lim_{a\to 0^+} \left(-1 + \frac{1}{a}\right) = \infty$$

diverges

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx \;=\; \lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx$



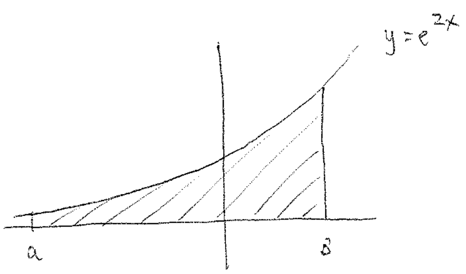
$$= \lim_{a\to 0^+} 2x^{1/2}\Big|_a^1$$

$$= \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2$$

converges

Example 3. Evaluate each of the following.

(a) $\displaystyle\int_{-\infty}^3 e^{2x}\,dx \;=\; \lim_{a\to -\infty} \int_a^3 e^{2x}\,dx$



$$= \lim_{a\to -\infty} \frac{1}{2} e^{2x}\Big|_a^3$$

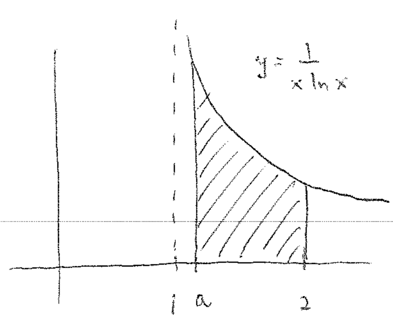
$$= \lim_{a\to -\infty} \frac{1}{2}\left(e^6 - e^{2a}\right) = \frac{1}{2}e^6$$

converges

(b) $\displaystyle\int_1^2 \frac{dx}{x\ln x} \;=\; \lim_{a\to 1^+} \int_a^2 \frac{dx}{x\ln x}$

$u = \ln x$

$du = \frac{1}{x}\,dx$



$$= \lim_{a\to 1^+} \int_{\ln a}^{\ln 2} \frac{du}{u}$$

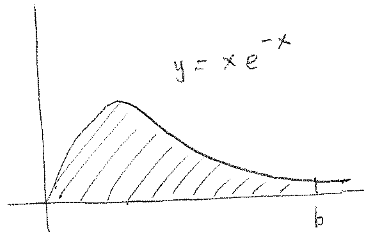
$$= \lim_{a\to 1^+} \ln|u|\Big|_{\ln a}^{\ln 2}$$

$$= \lim_{a\to 1^+} \left(\ln(\ln 2) - \ln(\ln a)\right) = \infty$$

diverges

(c) $\displaystyle\int_0^\infty xe^{-x}\,dx \quad = \lim_{b\to\infty} \int_0^b x e^{-x}\,dx$

IBP: $u = x$, $dv = e^{-x}\,dx$; $du = dx$, $v = -e^{-x}$



$$= \lim_{b\to\infty} \left[-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx \right]$$

$$= \lim_{b\to\infty} \left(-be^{-b} - e^{-x}\Big|_0^b \right)$$

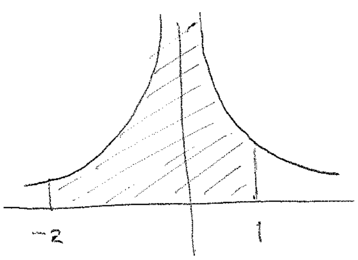
$$= \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1 \right) = 1$$

converges

Note: $\displaystyle\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \frac{b}{e^b} \underset{L'H}{=} \lim_{b\to\infty} \frac{1}{e^b} = 0$

If both limits are infinite or if there is a discontinuity in the interior of the interval, then we must split the integral into two improper integrals.

Example 4. Evaluate $\displaystyle\int_{-2}^1 \frac{1}{x^4}\,dx$.



We have $\displaystyle\int_0^1 \frac{1}{x^4} = \lim_{a\to 0^+} \int_a^1 x^{-4}\,dx$

$$= \lim_{a\to 0^+} -\frac{1}{3x^3}\Big|_a^1$$

$$= \lim_{a\to 0^+} \left(-\frac{1}{3} + \frac{1}{3a^3} \right) = \infty$$

Since this part diverges, the entire integral diverges.

By generalizing the calculations in Examples 1, 2, and 4, we obtain the following useful facts:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if } p > 1 \text{ and diverges if } p \le 1$$

$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges if } p < 1 \text{ and diverges if } p \ge 1$$

In many cases, it is easy to determine whether an improper integral converges or diverges even if we do not know how to evaluate it exactly. The idea is to compare it with a simpler integral that we do understand–such as one of the "p-integrals" above.

The Comparison Theorem. Suppose that $0 \le f(x) \le g(x)$ for $x \ge a$.

If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges

If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges

Example 5. Determine whether each of the following integrals converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^3+1}\,dx$

We have $\sin^2 x \le 1$ and $x^3+1 \ge x^3$, so $\dfrac{\sin^2 x}{x^3+1} \le \dfrac{1}{x^3}$. Hence the integral converges by direct comparison with the integral $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ $(p = 3 > 1)$.

(b) $\displaystyle\int_2^\infty \frac{1+e^{-x}}{\sqrt{x^2-3}}\,dx$

We have $1+e^{-x} \ge 1$ and $\sqrt{x^2-3} \le \sqrt{x^2} = x$, so

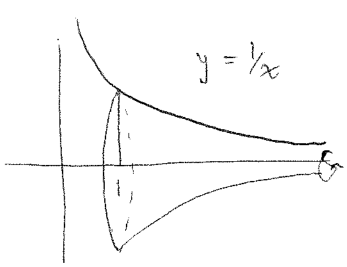
$\dfrac{1+e^{-x}}{\sqrt{x^2-3}} \ge \dfrac{1}{x}$. Hence the integral diverges by comparison with $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ $(p=1)$.

(c) $\displaystyle\int_1^\infty e^{-x^2}\,dx$

We have $e^{-x^2} \le e^{-x}$ for $x \ge 1$, and $\displaystyle\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}(-e^{-b}+1) = 1$.

Hence the integral converges by comparison with $\displaystyle\int_1^\infty e^{-x}\,dx$.

Example 6. **(Gabriel's Horn)** A solid is generated by revolving the infinite region bounded by the curve $y = 1/x$ and the x-axis for $x \ge 1$ about the x-axis. Show that this object has finite volume but infinite surface area.



By the disk method,

$$V = \lim_{b\to\infty}\int_1^b \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \pi \quad \text{by Example 1(a)}.$$

However,

$f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$$S = \lim_{b\to\infty}\int_1^b 2\pi\cdot\frac{1}{x}\cdot\sqrt{1+\frac{1}{x^4}}\,dx$$

$$\ge 2\pi\int_1^\infty \frac{1}{x}\,dx = \infty \quad \text{by Example 1(b)}.$$

§10.1—Sequences

An infinite sequence is a function whose domain is the positive integers. For example, the formula $a_n = 1/n$ defines a sequence whose values are

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad a_5 = \frac{1}{5}, \ldots.$$

In this case, we have $\lim_{n\to\infty} a_n = 0$, so we say that the sequence $\{a_n\}$ converges to 0.

Example 1. Find a formula for the nth term of each of the following sequences and determine the limits of the convergent sequences.

(a) $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots\}$

$$a_n = \frac{n}{n+1}$$

$$\lim_{n\to\infty} a_n = 1$$ converges

(b) $\{b_n\} = \{2, 5, 10, 17, 26, 37, 50, \ldots\}$

$$b_n = n^2 + 1$$

$$\lim_{n\to\infty} b_n = \infty$$ diverges

(c) $\{c_n\} = \{-1, 1, -1, 1, -1, 1, \ldots\}$

$$c_n = (-1)^n$$

$\lim_{n\to\infty} c_n$ does not exist

diverges

(d) $\{d_n\} = \{-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \frac{1}{729}, \ldots\}$

$$d_n = \left(-\frac{1}{3}\right)^n$$

$$\lim_{n\to\infty} d_n = 0$$ converges

Example 2. Find the limits of the following sequences.

(a) $a_n = \dfrac{n^2+1}{3n^2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{3n^2}\right) = \frac{1}{3}$$

(b) $b_n = \ln(2n+1) - \ln n$ "$\infty - \infty$"

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{2n+1}{n}\right) = \lim_{n\to\infty} \ln\left(2 + \frac{1}{n}\right) = \ln 2,$$

(c) $c_n = \dfrac{\cos n}{n}$ We have $-1 \le \cos n \le 1$ for all n

so $$\underset{\substack{\downarrow \\ 0}}{-\frac{1}{n}} \le \frac{\cos n}{n} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}.$$

$$\lim_{n\to\infty} c_n = 0$$

by Squeeze Theorem

Just as for functions of a real variable, limits of sequences that take the form "0/0" or "∞/∞" may be evaluated using L'Hôpital's Rule.

Example 3. Find the limits of the following sequences.

(a) $a_n = \dfrac{\ln n}{\sqrt{n}}$ "$\frac{\infty}{\infty}$"

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

(b) $b_n = n^{1/n}$ "∞^0"

$$\lim_{n\to\infty} \ln b_n = \lim_{n\to\infty} \frac{1}{n} \ln n = \lim_{n\to\infty} \frac{\ln n}{n} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

$$\underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

$$\therefore \lim_{n\to\infty} b_n = e^0 = 1$$

Many sequences of interest involve **factorials**. We write $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ for the product of the first n positive integers and read this as "n factorial".

Example 4. Find the limits of the following sequences.

(a) $a_n = \dfrac{3^n}{n!}$

We have
$$a_n = \frac{3\cdot 3\cdot 3\cdot \cdots \cdot 3\cdot 3\cdot 3}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{3}{n}\cdot \underbrace{\left(\frac{3}{n-1}\cdot\frac{3}{n-2}\cdot\cdots\cdot\frac{3}{5}\cdot\frac{3}{4}\right)}_{<1}\cdot 1\cdot\frac{3}{2}\cdot 3$$

$$< \frac{27}{2n}$$

Thus
$$\lim_{n\to\infty} a_n = 0.$$

(b) $b_n = \dfrac{n^2(2n-1)!}{(2n+1)!}$

$$= \frac{n^2(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}{(2n+1)(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{n^2}{(2n+1)(2n)}$$

$$= \frac{n^2}{4n^2+2n}$$

Note
$$\lim_{n\to\infty} \frac{n^2}{4n^2+2n} \underset{L'H}{=} \lim_{n\to\infty} \frac{2n}{8n+2} \underset{L'H}{=} \lim_{n\to\infty} \frac{2}{8} = \frac{1}{4}$$

Thus $\lim_{n\to\infty} b_n = \frac{1}{4}$.

Example 5. Find the limit of the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$. "$1^\infty$"

We have

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) \quad \text{"}\infty\cdot 0\text{"}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \quad \text{"}\frac{0}{0}\text{"}$$

$$\underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

Thus $\lim_{n\to\infty} a_n = e^1 = e$

§10.2—Summing an Infinite Series

An infinite series is the sum of an infinite sequence of numbers:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots.$$

To make sense of an infinite series, we look at the **sequence of partial sums**, defined by

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N.$$

If $\lim_{N\to\infty} S_N = S$, then we say that the series **converges** to S, and we refer to S as the sum of the infinite series. If the limit of partial sums does not exist, then the series **diverges**.

The series-integral analogy. Notice that this is the same terminology we used when studying improper integrals. Conceptually, the partial sums S_N play the role of the "partial integrals" $\int_1^b f(x)\,dx$. The idea is once again to deal with a finite problem first and then introduce a limit to handle the infinite problem.

Changing the starting point. It is not necessary that an infinite series start with the $n = 1$ term. Depending on the formula for a_n, it may be convenient to start at $n = 0$, $n = 2$, or some other value, say $n = k$. Thus in general we may consider series of the shape

$$\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \ldots$$

Here the Nth partial sum is defined to be the sum of the terms up to and including a_N.

A version of Zeno's dichotomy paradox states that one cannot get from point A to point B because one must first cover half the remaining distance, then half of the new remaining distance, *etc.* This creates an infinite process that appears to have no conclusion. Fortunately, we can use the theory of infinite series to resolve this issue:

Example 1. Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots.$

The Nth partial sum is

$$S_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \frac{1}{2^{N-1}} + \frac{1}{2^N}.$$

Multiplying through by $\frac{1}{2}$ gives

$$\frac{1}{2}S_N = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^N} + \frac{1}{2^{N+1}}$$

We now subtract the two equations to get

$$S_N - \frac{1}{2}S_N = \frac{1}{2} - \frac{1}{2^{N+1}}$$

$$\Rightarrow \frac{1}{2}S_N = \frac{1}{2} - \frac{1}{2^{N+1}}$$

$$\Rightarrow S_N = 1 - \frac{1}{2^N} \to 1 \text{ as } N \to \infty.$$

Thus $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1-\frac{1}{2^N}\right) = 1.$

The type of series encountered in Example 1 is called a **geometric series**. In general, a geometric series has the form

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \ldots.$$

Here c is the first term, and r is the common ratio between successive terms. By proceeding as in Example 1, we can find a formula for the Nth partial sum, S_N. We have

$$S_N = c + cr + cr^2 + \cdots + cr^{N-1} + cr^N \quad \text{and} \quad rS_N = cr + cr^2 + cr^3 + \cdots + cr^N + cr^{N+1},$$

so subtracting gives $(1-r)S_N = c - cr^{N+1}$, and thus

$$S_N = \frac{c - cr^{N+1}}{1-r},$$

provided that $r \neq 1$. Notice that if $|r| < 1$ then r^{N+1} approaches 0 as $N \to \infty$, so we have

$$\sum_{n=0}^{\infty} cr^n = \lim_{N\to\infty} S_N = \frac{c}{1-r} \qquad (|r| < 1).$$

The series diverges if $|r| \geq 1$.

Example 2. Find the sum of each series if it converges.

(a) $\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n} = \frac{3^3}{5} + \frac{3^4}{5^2} + \frac{3^5}{5^3} + \frac{3^6}{5^4} + \cdots$

geometric

$c = \frac{27}{5}$

$r = \frac{3}{5}$

$= \frac{27/5}{1 - 3/5}$

$= \frac{27/5}{2/5} = \frac{27}{2}$

(b) $\sum_{n=0}^{\infty} \frac{e^n}{2^{n+1}} = \frac{1}{2} + \frac{e}{2^2} + \frac{e^2}{2^3} + \frac{e^3}{2^4} + \cdots$

geometric

$c = \frac{1}{2}$

diverges since $|r| > 1$

$r = \frac{e}{2} \approx 1.36$

Example 3. Express each repeating decimal as a fraction.

(a) $.\overline{9} = .9 + .09 + .009 + .0009 + \cdots$

geometric

$= \frac{0.9}{1 - 0.1}$

$c = 0.9$

$r = 0.1$

$= \frac{0.9}{0.9} = 1$

(b) $.\overline{15} = .15 + .0015 + .000015 + \cdots$

geometric

$= \frac{0.15}{1 - 0.01}$

$c = 0.15$

$r = 0.01$

$= \frac{0.15}{0.99} = \frac{15}{99}$

Example 4. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots$.

Partial fractions: $\dfrac{1}{n(n+1)} = \dfrac{A}{n} + \dfrac{B}{n+1}$

$\Rightarrow 1 = A(n+1) + Bn$

$n = 0 \Rightarrow A = 1$

$n = -1 \Rightarrow B = -1$

So $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

Nth partial sum:

$$S_N = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(N-1)N} + \frac{1}{N(N+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

<u>telescoping</u>

$$= 1 - \frac{1}{N+1}$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1$

The Divergence Test. It is fairly obvious that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then we must have $\displaystyle\lim_{n\to\infty} a_n = 0$. Therefore, if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, we may conclude that the series diverges.

WARNING: The converse of this statement is false!

For example, $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

Example 5. Let $a_n = \dfrac{4+3n^2}{6n^2-2}$. Evaluate

(a) $\displaystyle\lim_{n\to\infty} a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4+3n^2}{6n^2-2} \underset{L'H}{=} \lim_{n\to\infty} \frac{6n}{12n} = \frac{1}{2}$$

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$

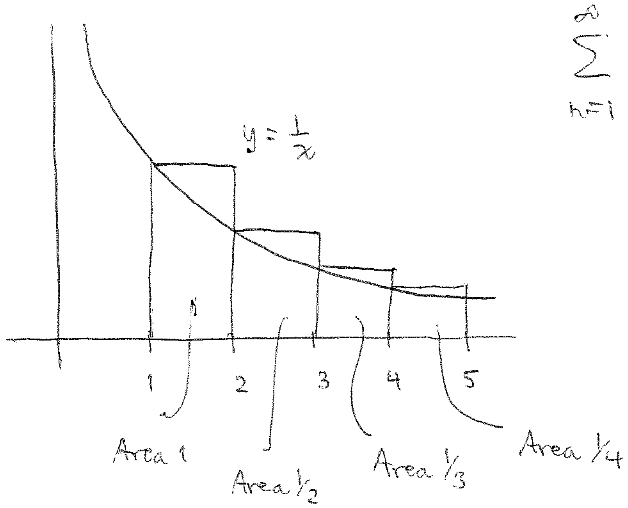
Since $\displaystyle\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$,

the series diverges by the Divergence Test.

In this case, $\displaystyle\sum_{n=1}^{\infty} a_n = \infty$.

§10.3—Convergence of Series with Positive Terms

Example 1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.



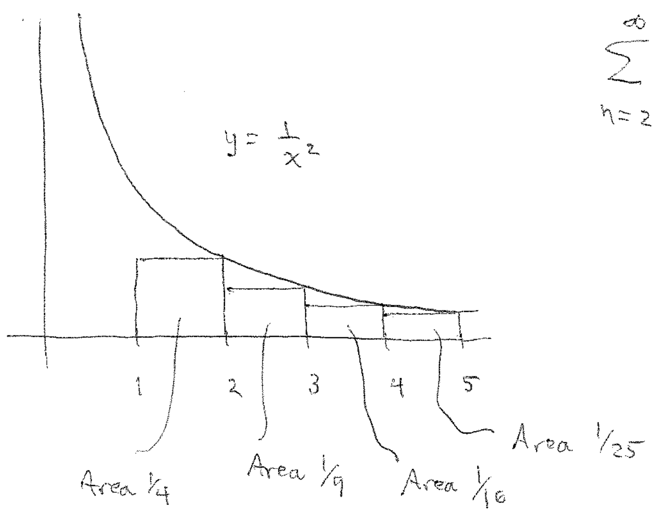
$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \ln b = \infty$$

Hence the series diverges.

Example 2. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges.



$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right) = 1$$

Hence the series converges.

The Integral Test. If $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, then the series $\sum_{n=1}^{\infty} f(n)$ and the integral $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

The p-series test. By taking $f(x) = \frac{1}{x^p}$ in the Integral Test and using the results of §7.7, we find that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Example 3. Determine whether the following series converge.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$

Consider $\int_1^{\infty} xe^{-x^2}\,dx$

$$= \lim_{b\to\infty} \int_1^b xe^{-x^2}\,dx \qquad u = -x^2, \quad du = -2x\,dx$$

$$= \lim_{b\to\infty} \int_{-1}^{-b^2} -\frac{1}{2}e^u\,du$$

$$= \lim_{b\to\infty} \left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}\right) = \frac{1}{2e}$$

Therefore the series converges by the Integral Test.

(b) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Consider $\int_2^{\infty} \frac{1}{x\ln x}\,dx$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx \qquad u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \left(\ln(\ln b) - \ln(\ln 2)\right) = \infty$$

Therefore the series diverges by the Integral Test.

Just as we saw for improper integrals in §7.7, we can often determine whether a series converges by comparing it with a simpler series that we already understand. The two most useful series for comparisons are:

1. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which converges for $p > 1$ and diverges for $p \le 1$
2. The geometric series $\sum_{n=0}^{\infty} cr^n$, which converges for $|r| < 1$ and diverges for $|r| \ge 1$

The Direct Comparison Test. Suppose that $0 \le a_n \le b_n$ for n sufficiently large.

(i) If $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Example 4. Determine whether the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$

We have $\frac{1}{n3^n} \leq \frac{1}{3^n}$, and we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series w/ $r = 1/3$).

Therefore the series converges by the Direct Comparison Test.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}}$

We have $\frac{\ln n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$, and we know that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series w/ $p = 1/2$).

Therefore the series diverges by the Direct Comparison Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

We have $\frac{1+\cos n}{n^2} \leq \frac{2}{n^2}$, and we know that $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series w/ $p = 2$). Therefore the series converges by the Direct Comparison Test.

Growth Rates of Sequences. Recall that two positive sequences a_n and b_n grow at the same rate if

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty.$$

If the limit is 0, then a_n grows slower than b_n. If the limit is ∞, then a_n grows faster than b_n. Since the convergence or divergence of a series depends only on the long-term behavior, we can replace the inequalities $\leq$ and $\geq$ in the Direct Comparison Test by the notions of "grows no faster than" and "grows no slower than". This yields the following test, which is often more powerful.

The Limit Comparison Test. Suppose that $a_n, b_n > 0$ and let

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

(i) If $0 < L < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.
(ii) If $L = \infty$ and $\sum a_n$ converges then $\sum b_n$ converges.
(iii) If $L = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.

Example 5. Determine whether the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^7+1}}$ $\quad a_n = \dfrac{n^2+1}{\sqrt{n^7+1}} \quad b_n = \dfrac{n^2}{\sqrt{n^7}} = \dfrac{n^2}{n^{7/2}} = \dfrac{1}{n^{3/2}}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^7+1}} \cdot \frac{\sqrt{n^7}}{n^2} = \lim_{n\to\infty} \frac{1+1/n^2}{\sqrt{1+1/n^7}} = 1$$

Since $\sum b_n$ converges (p-series, $p = 3/2$), $\sum a_n$ converges by the LCT part (i).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ $\quad a_n = \dfrac{\ln n}{n^3} \quad b_n = \dfrac{1}{n^2}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^3} \cdot n^2 = \lim_{n\to\infty} \frac{\ln n}{n} \stackrel{L'H}{=} 0$$

Since $\sum b_n$ converges (p-series, $p = 2$), $\sum a_n$ converges by the LCT part (iii).

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ $\quad a_n = \dfrac{1}{(\ln n)^2} \quad b_n = \dfrac{1}{n}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{(\ln n)^2} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{1}{2(\ln n)\cdot\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{2\ln n} = \infty$$

Since $\sum b_n$ diverges (p-series, $p=1$), $\sum a_n$ diverges by the LCT part (ii).

§10.4—Absolute and Conditional Convergence

We know that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \dots$$

diverges, since it is a p-series with $p = 1$. However, the *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n+1}}{n} + \dots$$

converges. This is a consequence of the following theorem.

The Alternating Series Test. Suppose that

$$a_1 > a_2 > a_3 > \dots > 0 \quad \text{and} \quad \lim_{n\to\infty} a_n = 0.$$

Then the series

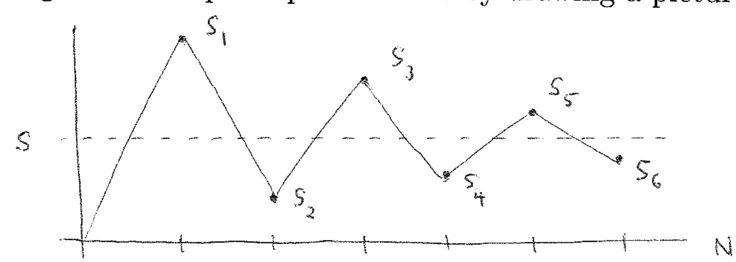
$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

converges. Furthermore, if S is the sum of the series and S_N is the Nth partial sum, then

$$|S - S_N| < a_{N+1}.$$

That is, the error in truncating the series is at most the size of the first unused term.

It is easy to give a conceptual proof of this by drawing a picture:



Example 1. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges, and determine the maximum error in approximating the sum using the first 10 terms.

Let $a_n = \frac{1}{n}$. Then $1 > \frac{1}{2} > \frac{1}{3} > \dots > 0$

and $\lim_{n\to\infty} \frac{1}{n} = 0$, so the series converges

by the AST.

Moreover, $|S - S_{10}| < a_{11} = \frac{1}{11}$

Example 2. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$ converges, and determine the maximum error in approximating the sum using the first 5 terms.

Let $a_n = \frac{1}{n^2}$. Then $1 > \frac{1}{4} > \frac{1}{9} > \ldots$ and

$\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the series converges by the AST.

Moreover, $|S - S_5| < a_6 = \frac{1}{36}$.

We say that the series $\sum a_n$ **converges absolutely** if the series $\sum |a_n|$ converges. A series that converges but does not converge absolutely is said to **converge conditionally**. For instance, the series in Example 1 converges conditionally, while the series in Example 2 converges absolutely.

It is easy to see that absolute convergence implies convergence. This observation can be useful for series with both positive and negative terms, even when the alternating series test doesn't apply.

Example 3. Determine whether each of the following series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ Let $a_n = \frac{1}{\sqrt{n}}$. Then $1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \ldots$

and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so the series converges by AST.

However, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series, $p = 1/2$), so the series converges conditionally.

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$ We have $\left| \frac{\sin n}{3^n} \right| \leq \frac{1}{3^n}$ and

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series, $r = 1/3$).

Thus $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$ converges absolutely by the Direct Comparison Test.

(c) $\sum_{n=1}^{\infty} (-1)^{n-1} \ln n$ We have $\lim_{n\to\infty} \ln n = \infty$, so the

series diverges by the Test for Divergence.

§10.5—The Ratio and Root Tests

Recall that $\sum a_n$ is a geometric series if there is a number r such that $a_{n+1}/a_n = r$ for each n. Furthermore, the series converges if and only if $|r| < 1$. More generally, if the ratios of successive terms of a series *approach* a common value, then the series behaves much like a geometric series in the long run, and we get a similar conclusion.

The Ratio Test. Suppose that the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

(a) If $\rho < 1$, then the series $\sum a_n$ converges absolutely.

(b) If $\rho > 1$, then the series $\sum a_n$ diverges.

(c) If $\rho = 1$, then the test gives no information.

Example 1. Use the Ratio Test to determine whether the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^2 5^n}$ $\qquad a_n = \frac{e^{2n}}{n^2 5^n} \qquad a_{n+1} = \frac{e^{2n+2}}{(n+1)^2 5^{n+1}}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{2n+2}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{e^{2n}}$$

$$= \lim_{n\to\infty} \frac{e^2}{5} \cdot \left(\frac{n}{n+1}\right)^2 = \frac{e^2}{5} > 1$$

Hence the series diverges by the Ratio Test.

(b) $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$ $\qquad a_n = \frac{3^n (n!)^2}{(2n)!} \qquad a_{n+1} = \frac{3^{n+1}((n+1)!)^2}{(2n+2)!}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{3^n (n!)^2}$$

$$= \lim_{n\to\infty} 3 \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} 3(n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{3(n^2+2n+1)}{4n^2+6n+2} = \frac{3}{4} < 1$$

Hence the series converges by the Ratio Test

The Ratio Test is particularly useful for series involving exponential functions (*e.g.* 2^n) and/or factorials. Series that only involve power functions (*e.g.* n^3 or $\sqrt{n}$) and logarithms will give $\rho = 1$, so a different test must be applied (usually comparison with a p-series).

The following test is applicable in similar situations but actually turns out to be slightly more powerful than the Ratio Test.

The Root Test. Suppose that the limit $L = \lim_{n\to\infty} |a_n|^{1/n}$ exists.

(a) If $L < 1$, then the series $\sum a_n$ converges absolutely.

(b) If $L > 1$, then the series $\sum a_n$ diverges.

(c) If $L = 1$, then the test gives no information.

Notice that both the Ratio Test and the Root Test are tests for absolute convergence–they cannot be used to detect conditional convergence.

Example 2. Use the Root Test to determine whether the series $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n+2}\right)^n$ converges.

We have
$$L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{4n+3}{3n+2} = \frac{4}{3} > 1.$$

Hence the series diverges by the Root Test.

Example 3. Let $a_n = \begin{cases} n2^{-n} & \text{for } n \text{ odd} \\ 2^{-n} & \text{for } n \text{ even} \end{cases}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

We have
$$|a_n|^{1/n} = \begin{cases} \frac{1}{2} n^{1/n} & \text{for } n \text{ odd} \\ \frac{1}{2} & \text{for } n \text{ even} \end{cases}$$

Since $\lim_{n\to\infty} n^{1/n} = 1$ (§10.1 notes, Ex 3b),

we see that $L = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1$, so

the series converges by the Root test.

Note that the Ratio Test is inconclusive because

$$\frac{a_{n+1}}{a_n} = \begin{cases} 1/2n & \text{for } n \text{ odd} \\ (n+1)/2 & \text{for } n \text{ even} \end{cases}$$
so $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

§10.6—Power Series

A **power series** centered at c has the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots.$$

Here x is a variable and $a_0, a_1, a_2, \dots$ are constants. The series may converge for some values of x and diverge for other values.

Example 1. For what values of x does the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converge?

First apply Ratio Test :

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n\to\infty} |x| \sqrt{\frac{n}{n+1}} = |x|$$

So the series converges if $|x| < 1$ and diverges if $|x| > 1$.

Now test the endpoints :

$\underline{x=1}$; $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series ($p = 1/2$)

$\underline{x=-1}$; $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST

Hence the series converges for $-1 \le x < 1$ and diverges otherwise.

Example 2. For what values of x does the power series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n^2 5^n}$ converge?

Ratio Test :

$$\rho = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{(n+1)^2\, 5^{n+1}} \cdot \frac{n^2\, 5^n}{(x-4)^n} \right|$$

$$= \lim_{n\to\infty} |x-4| \cdot \frac{1}{5} \left(\frac{n}{n+1}\right)^2 = \frac{|x-4|}{5}$$

so the series converges if $|x-4| < 5$ and diverges if $|x-4| > 5$.

Endpoints :

$\underline{x=9}$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent p-series ($p=2$)

$\underline{x=-1}$; $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by AST (or by absolute convergence)

Hence the series converges for $-1 \le x \le 9$ and diverges otherwise.

If the power series $\sum a_n(x-c)^n$ converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$, then R is called the **radius of convergence** of the series. The interval of x values for which the series converges is called the **interval of convergence**; it could be $(c-R, c+R)$, $[c-R, c+R)$, $(c-R, c+R]$, or $[c-R, c+R]$ depending on what happens at the endpoints. In Example 1, we have $R = 1$ and $I = [-1, 1)$. In Example 2, we have $R = 5$ and $I = [-1, 9]$.

Example 3. Find the radius and interval of convergence for each of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+1)^n}{2^n}$

Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x+1)^n} \right|$$

$$= \lim_{n\to\infty} |x+1| \cdot \frac{1}{2}\left(\frac{n+1}{n}\right) = \frac{|x+1|}{2} \qquad \text{So } R = 2$$

Endpoints:

$x = 1$: $\sum_{n=0}^{\infty} n$ diverges by Div. Test $\qquad$ So $I = (-3, 1)$

$x = -3$: $\sum_{n=0}^{\infty} (-1)^n n$ diverges by Div. Test

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \quad \text{for all } x$$

Thus $R = \infty$ and $I = (-\infty, \infty)$

(c) $\displaystyle\sum_{n=0}^{\infty} n!(x+3)^n$

Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+3)^{n+1}}{n!\,(x+3)^n} \right|$$

$$= \lim_{n\to\infty} (n+1)|x+3| = \begin{cases} \infty & \text{if } x \neq -3 \\ 0 & \text{if } x = -3 \end{cases}$$

Thus $R = 0$ and $I = \{-3\}$

§10.7—Taylor Series

When a power series converges, the resulting sum is a function of x. For example, the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

has first term 1 and common ratio x, so it converges to the function $f(x) = \frac{1}{1-x}$ whenever $|x| < 1$. We want to learn to generate power series for other interesting functions, and the following example shows that we get some additional mileage out of the geometric series.

Example 1. Find power series representations (valid when $|x| < 1$) for each of the following functions.

(a) $g(x) = \dfrac{1}{1+x^2}$

$$= f(-x^2)$$
$$= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots$$
$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for } |x| < 1$$

(b) $h(x) = \tan^{-1} x$

$$= \int g(x)\,dx$$
$$= \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for } |x| < 1$$

Goal: Given $f(x)$, try to choose coefficients a_n such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

Setting $x = 0$ gives $a_0 = f(0)$.

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \Rightarrow a_1 = f'(0)$

$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \Rightarrow a_2 = \frac{1}{2} f''(0)$

$f'''(x) = 6a_3 + 24a_4 x + \cdots \Rightarrow a_3 = \frac{1}{6} f'''(0)$

$f^{(4)}(x) = 24a_4 + \cdots \Rightarrow a_4 = \frac{1}{24} f^{(4)}(0)$

Note that $y = a_0 + a_1 x = f(0) + f'(0)x$ is just the tangent line to f at $x = 0$.

After some calculation, we find that $a_n = \dfrac{f^{(n)}(0)}{n!}$. Note that by convention $0! = 1$. The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$

is called the **Maclaurin series** for $f(x)$. In Example 1, we found two special Maclaurin series using ad-hoc arguments, but the above formula provides a systematic method.

Example 2. Compute the Maclaurin series for each of the following functions.

(a) $f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$ (In general, $f^{(n)}(0) = 1$)

$f''(x) = e^x \Rightarrow f''(0) = 1$

So $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

valid for all x

(b) $f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$

So $\sin x = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040} + \dots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

valid for all x

(c) $f(x) = \cos x$

Can use method of part (b) or the following shortcut:

$$\begin{aligned} \cos x &= \frac{d}{dx}(\sin x) \\ &= \frac{d}{dx}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots\right) \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

valid for all x

More generally, the series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + \dots$$

is called the **Taylor series** for $f(x)$ centered at $x = c$. Note that the Maclaurin series is just another name for the Taylor series centered at $x = 0$.

Example 3. Find the Taylor series for $f(x) = \ln x$ about $x = 1$.

$f(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4} \Rightarrow f^{(4)}(1) = -6$

$f^{(5)}(x) = \frac{24}{x^5} \Rightarrow f^{(5)}(1) = 24$

In general,

$f^{(n)}(1) = (-1)^{n-1}(n-1)!$

for $n \geq 1$

Thus $\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$

valid for $0 < x \leq 2$

Taylor and Maclaurin series for functions whose series are already known can often be obtained by a simple manipulation of the original series.

Example 4. Find the Maclaurin series for each of the following functions by using the results of Example 2.

(a) $f(x) = e^{-x^2}$

$$= 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{6} + \frac{(-x^2)^4}{24} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

valid for all x

(b) $f(x) = x \cos 3x$

$$= x\left[1 - \frac{(3x)^2}{2} + \frac{(3x)^4}{24} - \frac{(3x)^6}{720} + \cdots\right]$$

$$= x - \frac{9}{2}x^3 + \frac{81}{24}x^5 - \frac{729}{720}x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-9)^n x^{2n+1}}{(2n)!}$$

valid for all x

The Binomial Series. When $|x| < 1$, we have

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{6}x^3 + \cdots + \binom{a}{n}x^n + \ldots,$$

where

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}.$$

Note that if a is a positive integer, then this is just a finite sum because $\binom{a}{n} = 0$ when $n > a$. In this case, we recover the familiar Binomial Theorem; for instance,

$$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7.$$

Example 5. Find the first four terms of the Maclaurin series for $f(x) = (1+x)^{-1/3}$.

$$(1+x)^{-1/3} = 1 + (-1/3)x + \binom{-1/3}{2}x^2 + \binom{-1/3}{3}x^3 + \dots$$

$$= 1 - \frac{1}{3}x + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2}x^2 + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{6}x^3 + \dots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \dots$$

for $|x| < 1$

Example 6. Use the first three terms of a series to estimate each integral and find upper bounds for the error in your estimates.

(a) $\displaystyle\int_0^{2/3} e^{-x^2}\,dx,$ Use Ex 4a :

$$= \int_0^{2/3} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots\right) dx$$

$$= x - \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{42} - \dots \Big|_0^{2/3}$$

$$= \frac{2}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^3 - \frac{1}{10}\left(\frac{2}{3}\right)^5 + \frac{1}{42}\left(\frac{2}{3}\right)^7 - \dots$$

$$\approx \frac{2}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^3 - \frac{1}{10}\left(\frac{2}{3}\right)^5$$

$$\approx 0.58107$$

Error $\le \frac{1}{42}\left(\frac{2}{3}\right)^7 \approx 0.0139$ by AST

(b) $\displaystyle\int_0^{1/2} (1+x^2)^{-1/3}\,dx$ Use Ex 5 with $x \to x^2$:

$$= \int_0^{1/2} \left(1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \dots\right) dx$$

$$= x - \frac{1}{9}x^3 + \frac{2}{45}x^5 - \frac{2}{81}x^7 + \dots \Big|_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{9}\left(\frac{1}{2}\right)^3 + \frac{2}{45}\left(\frac{1}{2}\right)^5 - \frac{2}{81}\left(\frac{1}{2}\right)^7 + \dots$$

$$\approx \frac{1}{2} - \frac{1}{9}\left(\frac{1}{2}\right)^3 + \frac{2}{45}\left(\frac{1}{2}\right)^5$$

$$= 0.4875$$

Error $\le \frac{2}{81}\left(\frac{1}{2}\right)^7 \approx 0.00019$ by AST

§8.4—Taylor Polynomials

We can get an approximation to the function $f(x)$ when x is close to some fixed number c by truncating the Taylor series after finitely many terms. This gives the **Taylor polynomial** of order n centered at $x = c$:

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Notice that $T_1(x) = f(c) + f'(c)(x-c)$ is the familiar linear (tangent line) approximation.

Example 1. Find the Taylor polynomial of order 2 for $f(x) = \sqrt[3]{x}$ centered at $c = 8$. Then use it to estimate $\sqrt[3]{8.1}$, and find and upper bound for the error.

$f(x) = x^{1/3} \Rightarrow f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3} \Rightarrow f''(8) = -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3} \Rightarrow f'''(8) = \frac{10}{27 \cdot 256}$

Thus $T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2$

$= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

So $\sqrt[3]{8.1} = f(8.1) \approx T_2(8.1) = 2 + \frac{1}{12}(0.1) - \frac{1}{288}(0.1)^2$

≈ 2.008298611

By AST,

Error $\leq \frac{f'''(8)}{3!}(0.1)^3 \approx 2.41 \times 10^{-7}$

Example 2. Find the Taylor polynomial of order 2 for $f(x) = \cos x$ centered at $c = 0$. For what values of x will this polynomial approximate $\cos x$ with error at most 0.01?

$f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

So $T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$= 1 - \frac{1}{2}x^2$ (could also get this from §10.7 Ex 2c)

AST $\Rightarrow$ Error $\leq \frac{1}{24}x^4 \leq 0.01$

$\Leftrightarrow x^4 \leq 0.24$

$\Leftrightarrow |x| \leq (0.24)^{1/4} \approx 0.6999$

To estimate the error in a Taylor polynomial approximation when the series is not alternating, we often need the following result.

Taylor's Inequality. If $f^{(n+1)}$ is continuous and $|f^{(n+1)}(u)| \le K$ whenever u lies between c and x, then one has

$$|f(x) - T_n(x)| \le \frac{K}{(n+1)!}|x-c|^{n+1}.$$

Example 3. Find the Taylor polynomial of order 3 for $f(x) = \ln(1+2x)$ centered at $c = 1$, and estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $0.5 \le x \le 1.5$.

$$f(x) = \ln(1+2x) \Rightarrow f(1) = \ln 3$$

$$f'(x) = \frac{1}{1+2x} \cdot 2 \Rightarrow f'(1) = \frac{2}{3}$$

$$f''(x) = -2(1+2x)^{-2} \cdot 2 = \frac{-4}{(1+2x)^2} \Rightarrow f''(1) = -\frac{4}{9}$$

$$f'''(x) = 8(1+2x)^{-3} \cdot 2 = \frac{16}{(1+2x)^3} \Rightarrow f'''(1) = \frac{16}{27}$$

Thus $T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3$

Now $f^{(4)}(x) = -48(1+2x)^{-4} \cdot 2 \Rightarrow |f^{(4)}(u)| \le \frac{96}{(1+2(0.5))^4}$

when u lies between 0.5 and 1.5, so can take $K = \frac{96}{2^4} = 6$.

Hence

$$|f(x) - T_3(x)| \le \frac{6}{4!}(0.5)^4 = \frac{6}{24 \cdot 16} = \frac{1}{64} \quad \text{for } 0.5 \le x \le 1.5.$$

Example 4. Use a Maclaurin polynomial of order 5 to approximate $\sqrt{e}$, and find an upper bound for the error in your estimate.

From the Maclaurin series for e^x, we immediately get

$$T_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

and hence $\sqrt{e} = e^{1/2} \approx T_5(1/2)$

$$= 1 + \frac{1}{2} + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{6} + \frac{(1/2)^4}{24} + \frac{(1/2)^5}{120}$$

$$\approx 1.648697917$$

Moreover, $|f^{(6)}(u)| = e^u \le e^{1/2}$ when $0 \le u \le 1/2$, so

$$|\sqrt{e} - T_5(1/2)| \le \frac{e^{1/2}}{6!}\left(\frac{1}{2}\right)^6 < 3.58 \times 10^{-5}$$