MAT 161—Exam #3—11/18/14

Name: Solutions

Calculators are NOT allowed. Show all work using correct mathematical notation.

1. (20 points) Consider the function $f(x) = 3x^5 - 20x^3$.

   (a) Determine the intervals on which f is increasing/decreasing.

   (b) Determine the intervals on which f is concave up/concave down.

   (c) Sketch a graph of the function, clearly labeling the coordinates of all intercepts, local extrema, and inflection points.

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4)$$

f': $+$ | -2 | $-$ | 0 | $-$ | 2 | $+$

f: $\nearrow$ -2 $\searrow$ 0 $\searrow$ 2 $\nearrow$

$$f''(x) = 60x^3 - 120x = 60x(x^2 - 2)$$

f'': $-$ | $-\sqrt{2}$ | $+$ | 0 | $-$ | $\sqrt{2}$ | $+$

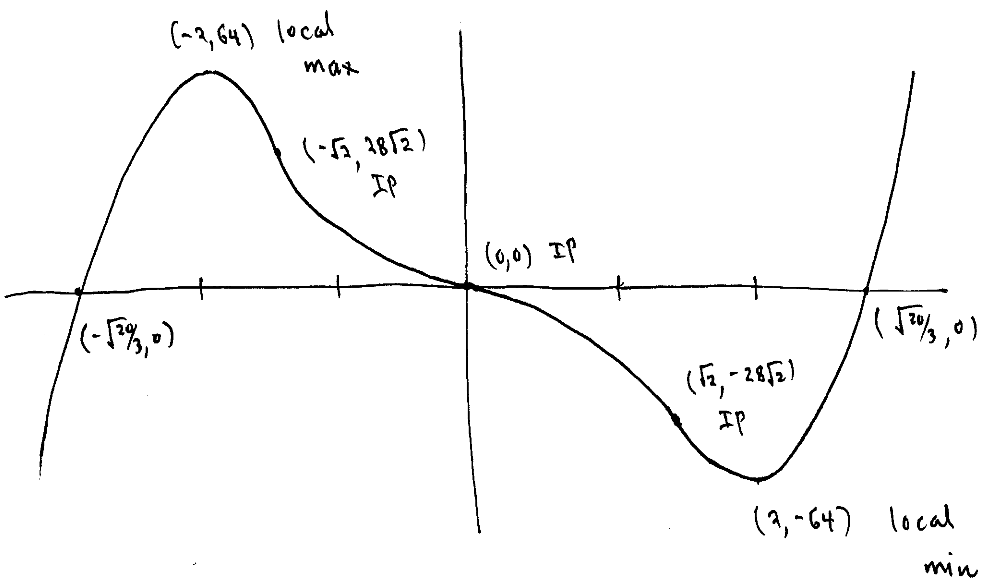
f: $\frown$ $-\sqrt{2}$ $\smile$ 0 $\frown$ $\sqrt{2}$ $\smile$

increasing: $(-\infty, -2)$, $(2, \infty)$

decreasing: $(-2, 2)$

concave up: $(-\sqrt{2}, 0)$, $(\sqrt{2}, \infty)$

concave down: $(-\infty, -\sqrt{2})$, $(0, \sqrt{2})$



2. (15 points) Evaluate each of the following limits. Show all work using correct notation!

(a) $\lim_{x\to\infty} \frac{\ln(5x+2)}{\ln(4x+3)}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{5x+2}\cdot 5}{\frac{1}{4x+3}\cdot 4} = \lim_{x\to\infty} \frac{5(4x+3)}{4(5x+2)}$

$= 1$ by leading coeffs or L'H again

(b) $\lim_{x\to 0} \frac{e^{3x}-1-3x}{1-\cos(5x)}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{3e^{3x}-3}{5\sin 5x}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{9e^{3x}}{25\cos 5x} = \frac{9}{25}$

3. (15 points) Evaluate each of the following indefinite integrals.

(a) $\int \left(2x^4+e^{-2x}+5\sec^2 x+7\right)\,dx$

$= \frac{2}{5}x^5 - \frac{1}{2}e^{-2x} + 5\tan x + 7x + C$

(b) $\int \left(\sqrt{x}+\frac{3}{x^2}-\frac{6}{x}+3\sin 5x\right)\,dx$

$= \frac{2}{3}x^{3/2} - \frac{3}{x} - 6\ln|x| - \frac{3}{5}\cos 5x + C$

4. (15 points) Consider the function $f(x) = \dfrac{1}{\sqrt{x}}$.

(a) Find the linearization $L(x)$ of $f(x)$ at $x = 4$.

$f(4) = \frac{1}{\sqrt{4}} = \frac{1}{2}$ $\qquad f'(x) = -\frac{1}{2}x^{-3/2}$

$\Rightarrow f'(4) = \frac{-1}{2 \cdot 4^{3/2}} = \frac{-1}{16}$

Thus $L(x) = \frac{1}{2} - \frac{1}{16}(x-4)$

(b) Use the linearization from part (a) to give an estimate for $\dfrac{1}{\sqrt{4.2}}$.

$\frac{1}{\sqrt{4.2}} = f(4.2) \approx L(4.2)$

$= \frac{1}{2} - \frac{1}{16}(0.2)$

$= \frac{1}{2} - \frac{1}{80} = \frac{39}{80}$

5. (10 points) Find the absolute maximum and minimum values of the function $f(x) = x \ln x$ on the interval $[e^{-2}, 1]$. *Hint:* Recall that $e > 2$.

$f'(x) = x \cdot \frac{1}{x} + \ln x$

$= 1 + \ln x = 0$

$\Rightarrow \ln x = -1$

$\Rightarrow x = e^{-1}$

Test crit pt & endpoints:

$f(e^{-2}) = e^{-2} \ln e^{-2}$ $\qquad f(e^{-1}) = e^{-1} \ln e^{-1}$ $\qquad f(1) = 0$

$= \frac{-2}{e^2}$ $\qquad = -\frac{1}{e}$

Since $e > 2$, $\frac{-2}{e^2} > \frac{-1}{e}$

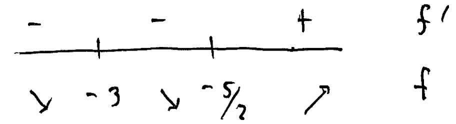
So 0 is the max

and $-\frac{1}{e}$ is the min.

6. (7 points) Find the interval(s) on which the function $f(x) = \dfrac{e^{2x}}{x+3}$ is increasing.

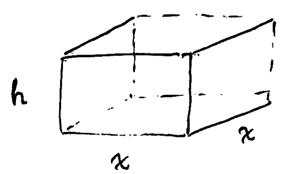
$$f'(x) = \frac{(x+3)\cdot 2e^{2x} - e^{2x}\cdot 1}{(x+3)^2}$$

$$= \frac{e^{2x}(2x+5)}{(x+3)^2}$$



f is increasing on $(-5/2, \infty)$.

7. (18 points) You are asked to design a box of volume 500 cm^3, with square base and no top. Find the dimensions that minimize the total amount of material used, and justify that your answer gives a minimum.



Want to minimize $S = x^2 + 4xh$,

subject to the constraint $x^2 h = 500$.

Since $h = 500/x^2$, we obtain

$$S(x) = x^2 + 4x\,(500/x^2)$$

$$= x^2 + \frac{2000}{x}$$

$$\Rightarrow \quad S'(x) = 2x - \frac{2000}{x^2} = 0$$

$$\Rightarrow \quad 2x^3 = 2000$$

$$\Rightarrow \quad x = \sqrt[3]{1000} = 10$$

$$\Rightarrow \quad h = 500/10^2 = 5$$

Note that $S''(x) = 2 + \dfrac{4000}{x^3} > 0$ for all $x > 0$, so the 2nd derivative test implies that the given solution is a minimum.