SOME FURTHER TRANSFORMATIONS FOR WP-BAILEY PAIRS

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Abstract. Let $(\alpha_n(a,k),\beta_n(a,k))$ be a WP-Bailey pair. Assuming the limits exist, let

$$(\alpha_n^*(a),\beta_n^*(a)) = \lim_{k\to 1}\left(\alpha_n(a,k),\frac{\beta_n(a,k)}{1-k}\right)$$

be the *derived* WP-Bailey pair. By considering a particular limiting case of a transformation due to George Andrews, we derive some transformation and summation formulae for derived WP-Bailey pairs.

We then use the formula to derive new identities for various theta series/products which can be expressed in terms of certain types of Lambert series and various other series-product identities.

1. Introduction

A *WP-Bailey pair* is a pair of sequences $(\alpha_n(a,k,q),\ \beta_n(a,k,q))$ satisfying $\alpha_0(a,k,q) = \beta_0(a,k,q) = 1$ and

$$\beta_n(a,k,q) = \sum_{j=0}^{n} \frac{(k/a;q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(aq;q)_{n+j}}\alpha_j(a,k,q). \tag{1.1}$$

In what follows, we define, for a WP-Bailey pair $(\alpha_n(a,k),\beta_n(a,k))$ and $n \geq 1$,

$$\begin{aligned}\alpha_n^* &= \alpha_n^*(a) = \alpha_n^*(a,q) := \lim_{k\to 1}\alpha_n(a,k),\\ \beta_n^* &= \beta_n^*(a) = \beta_n^*(a,q) = \lim_{k\to 1}\frac{\beta_n(a,k)}{1-k},\end{aligned} \tag{1.2}$$

assuming the limits exist. For ease of notation we call such a pair of sequences (α_n^*,β_n^*) a *derived* WP-Bailey pair. The main result of the present

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paper may be described as follows. Define

$$f_2(a,q) := \sum_{n=1}^{\infty} a^{2n}q^n\beta_n^*(a) - \sum_{n=1}^{\infty} a^{2n}q^n\beta_n^*(-a) + \sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(qa^2;q)_{2n}}a^{2n}q^n\alpha_n^*(-a) - \sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(qa^2;q)_{2n}}a^{2n}q^n\alpha_n^*(a). \tag{1.3}$$

Theorem 1. *Let $(\alpha_n(a,k),\beta_n(a,k))$ be a WP-Bailey pair and a and b complex numbers such that all of the derived WP-Bailey pairs (α_n^*,β_n^*) below exist. Let $f_2(a,q)$ be as defined at* (1.3) *and suppose further that each of series $f_2(a,q)$, $f_2(b,q)$, $f_2(1/a,q)$, $f_2(1/b,q)$ converges. Then*

$$\begin{aligned} &f_2(a,q) - f_2(b,q) - f_2(1/a,q) + f_2(1/b,q) \\ &\quad= \frac{2(a-b)(1+ab)}{(1-a^2)(1-b^2)} - 2a\frac{(b/a,qa/b,-ab,-q/ab;q)_\infty(q^2,q^2;q^2)_\infty}{(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}. \end{aligned} \tag{1.4}$$

We remark that one reason this result is of interest is that the right side is independent of the particular WP-Bailey pair $(\alpha_n(a,k),\beta_n(a,k))$ employed on the left side. We give some explicit examples below.

We apply this identity, and others proved below, to derive new series-product identities. For example, if a and b are non-zero complex numbers such that aq^n, $bq^n \neq \pm 1$, for $n \in \mathbb{Z}$, and $|q| < \max\{|a^2|, 1/|a^2|, |b^2|, 1/|b^2|\}$, then

$$\begin{aligned} \sum_{n=1}^{\infty}\Bigg[&\left(\frac{(1/a;q)_n}{(qa;q)_n} - \frac{(-1/a;q)_n}{(-qa;q)_n}\right)\frac{a^{2n}q^n}{1-q^n} \\ &- \left(\frac{(a;q)_n}{(q/a;q)_n} - \frac{(-a;q)_n}{(-q/a;q)_n}\right)\frac{q^n}{a^{2n}(1-q^n)} \\ &- \left(\frac{(1/b;q)_n}{(qb;q)_n} - \frac{(-1/b;q)_n}{(-qb;q)_n}\right)\frac{b^{2n}q^n}{1-q^n} \\ &+ \left(\frac{(b;q)_n}{(q/b;q)_n} - \frac{(-b;q)_n}{(-q/b;q)_n}\right)\frac{q^n}{b^{2n}(1-q^n)}\Bigg] \\ = \frac{2(a-b)(1+ab)}{(1-a^2)(1-b^2)} &- 2a\frac{(b/a,qa/b,-ab,-q/ab;q)_\infty(q^2,q^2;q^2)_\infty}{(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}. \end{aligned}$$

To set these results in perspective, we recall some of the history of WP-Bailey pairs, which were defined by Andrews in [1], where the concept of a *WP-Bailey chain* was also introduced. Andrews showed that if the pair $(\alpha_n(a,k),\ \beta_n(a,k))$ satisfies (1.1), then so does $(\hat{\alpha}_n(a,k),\ \hat{\beta}_n(a,k))$ where

$$\begin{aligned} \hat{\alpha}_n(a,k) &= \frac{(y,z;q)_n}{(aq/y,aq/z;q)_n}\left(\frac{k}{c}\right)^n\alpha_n(a,c), \\ \hat{\beta}_n(a,k) &= \frac{(ky/a,kz/a;q)_n}{(aq/y,aq/z;q)_n} \end{aligned} \tag{1.5}$$

$$\times \sum_{j=0}^{n} \frac{(1-cq^{2j})(y,z;q)_j(k/c;q)_{n-j}(k;q)_{n+j}}{(1-c)(ky/a,kz/a;q)_n(q;q)_{n-j}(qc;q)_{n+j}} \left(\frac{k}{c}\right)^j \beta_j(a,c),$$

with $c = kyz/aq$. Andrews also described a second method for deriving new WP-Bailey pairs from existing pairs. If $(\alpha_n(a,k),\, \beta_n(a,k))$ satisfy (1.1), then so does $(\tilde{\alpha}_n(a,k),\, \tilde{\beta}_n(a,k))$,

$$\begin{aligned}\tilde{\alpha}_n(a,k) &= \frac{(qa^2/k)_{2n}}{(k)_{2n}} \left(\frac{k^2}{qa^2}\right)^n \alpha_n\left(a, \frac{qa^2}{k}\right), \\ \tilde{\beta}_n(a,k) &= \sum_{j=0}^{n} \frac{(k^2/qa^2)_{n-j}}{(q)_{n-j}} \left(\frac{k^2}{qa^2}\right)^j \beta_j\left(a, \frac{qa^2}{k}\right).\end{aligned} \tag{1.6}$$

Each of these processes may be iterated, so that a single WP-Bailey pair gives rises to a *chain* of pairs, and all WP-Bailey chains together give rise to the *WP-Bailey tree* or *lattice*.

WP-Bailey chains were further investigated by Andrews and Berkovich [2], Spiridonov [13], Warnaar [14], Liu and Ma [8] and Mc Laughlin and Zimmer [10]. Previous work on WP-Bailey chains had appeared in the papers of Bressoud [7] and Singh [12] (although of course not using the terminology introduced by Andrews [1]).

In a recent paper [9], the present author investigated the implications of letting $y \to 1$ in the WP-Bailey chain at (1.5), and found, amongst other results, a number of new transformations relating WP-Bailey pairs, some similar transformations for standard Bailey pairs, and new expansions in terms of basic hypergeometric series for several theta products described by Ramanujan.

It is not difficult to show that (1.6) implies (see Corollary 1 in [11], for example) that if $(\alpha_n(a,k), \beta_n(a,k))$ satisfy (1.1), then subject to suitable convergence conditions,

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{qa^2}{k^2}\right)^n \beta_n(a,k) \\ = \frac{(qa/k, qa^2/k;q)_\infty}{(qa, qa^2/k^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(qa^2/k;q)_{2n}} \left(\frac{qa^2}{k^2}\right)^n \alpha_n(a,k).\end{aligned} \tag{1.7}$$

The results in the present paper are derived as consequences of letting $k \to 1$ in (1.7).

We employ the usual notations:

$$\begin{aligned}(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}), \\ (a_1,a_2,\dots,a_j;q)_n &:= (a_1;q)_n(a_2;q)_n\cdots(a_j;q)_n, \\ (a;q)_\infty &:= (1-a)(1-aq)(1-aq^2)\cdots,\ \text{and} \\ (a_1,a_2,\dots,a_j;q)_\infty &:= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_j;q)_\infty,\end{aligned}$$

We also make use of the q-Gauss sum

$$\sum_{n=0}^{\infty}\frac{(A,B;q)_n}{(C,q;q)_n}\left(\frac{C}{AB}\right)^n=\frac{(C/A,C/B;q)_\infty}{(C,C/AB;q)_\infty}. \tag{1.8}$$

Unless stated otherwise, we assume throughout that $|q|<1$.

2. Proof of the Main Identities

We now prove the main results.

Lemma 1. *Let $(\alpha_n(a,k),\ \beta_n(a,k))$ be a WP-Bailey pair and (α_n^*,β_n^*) the derived pair. For $|q|$, $|qa|$, $|qa^2|<1$ and assuming suitable convergence conditions,*

$$\sum_{n=1}^{\infty}a^{2n}q^n\beta_n^*(a)-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(qa^2;q)_{2n}}a^{2n}q^n\alpha_n^*(a)=f_1(a,q), \tag{2.1}$$

where

$$\begin{aligned} f_1(a,q)&=\sum_{n=1}^{\infty}\frac{(1/a;q)_n a^{2n}q^n}{(aq;q)_n(1-q^n)} \\ &=-\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(a,a;q)_n(q;q)_{2n-1}a^nq^n}{(1-a)(q,q;q)_n(qa^2;q)_{2n}} \\ &=\sum_{n=1}^{\infty}\frac{a^2q^n}{1-a^2q^n}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}. \end{aligned} \tag{2.2}$$

We remark that the right side in the above identity is independent of the particular derived pair (α_n^*,β_n^*) inserted on the left side. Also, for later use we note that

$$f_2(a,q)=f_1(a,q)-f_1(-a,q)=-\sum_{n=1}^{\infty}\frac{2aq^n}{1-a^2q^{2n}}, \tag{2.3}$$

where $f_2(a,q)$ is as defined at (1.3).

Proof of Lemma 1. Rewrite (1.7) as

(2.4)

$$\sum_{n=1}^{\infty}\left(\frac{qa^2}{k^2}\right)^n\frac{\beta_n(a,k)}{1-k}-\frac{\left(\frac{qa}{k},\frac{qa^2}{k};q\right)_\infty}{(qa,qa^2/k^2;q)_\infty}\sum_{n=1}^{\infty}\frac{(kq;q)_{2n-1}}{(qa^2/k;q)_{2n}}\left(\frac{qa^2}{k^2}\right)^n\alpha_n(a,k)$$
$$=\frac{1}{1-k}\left(\frac{(qa/k,qa^2/k;q)_\infty}{(qa,qa^2/k^2;q)_\infty}-1\right).$$

The left side of (2.1) now follows upon letting $k\to 1$. To get the first expression for $f_1(a,q)$, use (1.8) to expand the infinite product on the right side as an infinite series (set $A=k$, $B=k/a$ and $C=qa$) and then once again let $k\to 1$.

The second expression for $f_1(a,q)$ follows upon substituting the unit WP-Bailey pair

$$\begin{aligned}\alpha_n(a,k) &= \frac{(q\sqrt{a},-q\sqrt{a},a,a/k;q)_n}{(\sqrt{a},-\sqrt{a},q,kq;q)_n}\left(\frac{k}{a}\right)^n, \\ \beta_n(a,k) &= \begin{cases}1 & n=0,\\ 0, & n>0,\end{cases}\end{aligned} \tag{2.5}$$

into (1.2) and then inserting the resulting pair

$$\begin{aligned}\alpha_n(a,k) &= \frac{1-aq^{2n}}{1-a}\frac{(a,a;q)_n}{(q,q;q)_n}\left(\frac{1}{a}\right)^n, \\ \beta_n(a,k) &= 0\end{aligned} \tag{2.6}$$

on the left side of (2.1).

For the third representation of $f_1(a,q)$ define

$$G(k) := \frac{(qa/k,qa^2/k;q)_\infty}{(qa,qa^2/k^2;q)_\infty}$$

and then

$$\lim_{k\to 1}\frac{1}{1-k}\left(\frac{(qa/k,qa^2/k;q)_\infty}{(qa,qa^2/k^2;q)_\infty}-1\right) = \lim_{k\to 1}\frac{G(k)-G(1)}{1-k} = -G'(1),$$

and logarithmic differentiation now easily gives the result. □

Remark: The first expression for $f_1(a,q)$ above also follows from inserting the "trivial" WP-Bailey pair

$$\begin{aligned}\alpha_n(a,k) &= \begin{cases}1 & n=0,\\ 0, & n>0,\end{cases} \\ \beta_n(a,k) &= \frac{(k,k/a;q)_n}{(q,aq;q)_n},\end{aligned} \tag{2.7}$$

into (1.2) and then inserting the resulting derived pair

$$\begin{aligned}\alpha^*_n(a) &= 0, \\ \beta^*_n(a) &= \frac{(1/a;q)_n}{(aq;q)_n(1-q^n)}\end{aligned} \tag{2.8}$$

on the left side of (2.1).

One way of viewing Lemma 1 is as supplying a large number of representations of the difference of Lambert Series

$$\sum_{n=1}^\infty \frac{a^2q^n}{1-a^2q^n} - \sum_{n=1}^\infty \frac{aq^n}{1-aq^n}.$$

Indeed, such a representation arises if any pair (α^*_n,β^*_n) deriving from a WP-Bailey is inserted in (2.1), assuming the limits exist and the resulting series converge. We give two example below. The first arises from Singh's WP-Bailey pair [12]:

$$\alpha_n(a,k)=\frac{(q\sqrt{a},-q\sqrt{a},a,\rho_1,\rho_2,a^2q/k\rho_1\rho_2;q)_n}{(\sqrt{a},-\sqrt{a},q,aq/\rho_1,aq/\rho_2,k\rho_1\rho_2/a;q)_n}\left(\frac{k}{a}\right)^n, \tag{2.9}$$

$$\beta_n(a,k)=\frac{(k\rho_1/a,k\rho_2/a,k,aq/\rho_1\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2,k\rho_1\rho_2/a,q;q)_n}.$$

This gives the derived pair

$$\alpha^*_n(a)=\frac{(q\sqrt{a},-q\sqrt{a},a,\rho_1,\rho_2,a^2q/\rho_1\rho_2;q)_n}{(\sqrt{a},-\sqrt{a},q,aq/\rho_1,aq/\rho_2,\rho_1\rho_2/a;q)_n}\left(\frac{1}{a}\right)^n, \tag{2.10}$$

$$\beta^*_n(a)=\frac{(\rho_1/a,\rho_2/a,aq/\rho_1\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2,\rho_1\rho_2/a;q)_n(1-q^n)}.$$

The parameters ρ_1 and ρ_2 are free, but for simplicity we let $\rho_1,\rho_2\to\infty$ to get the derived WP-Bailey pair

$$\alpha^*_n(a)=\frac{1-aq^{2n}}{1-a}\frac{(a;q)_n}{(q;q)_n}(-1)^nq^{n(n-1)/2}, \tag{2.11}$$

$$\beta^*_n(a)=\frac{(-1)^nq^{n(n-1)/2}}{a^n}.$$

The WP-Bailey pair

$$\alpha_n(a,k)=\frac{\left(q\sqrt{a},-q\sqrt{a},a,a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}},\frac{a}{\sqrt{k}},-\frac{aq}{\sqrt{k}},\frac{k}{a};q\right)_n}{\left(\sqrt{a},-\sqrt{a},q,\sqrt{kq},-\sqrt{kq},q\sqrt{k},-\sqrt{k},\frac{qa^2}{k};q\right)_n}\left(\frac{k}{a}\right)^n, \tag{2.12}$$

$$\beta_n(a,k)=\frac{\left(\sqrt{k},\frac{k^2}{a^2};q\right)_n}{(q\sqrt{k},q;q)_n},$$

provides the derived pair

$$\alpha^*_n(a)=\frac{1-aq^{2n}}{1-a}\frac{\left(a,a,-aq,1/a;q)_n(a^2q;q^2\right)_n}{(q,q,-1,qa^2;q)_n(q;q^2)_n}\left(\frac{1}{a}\right)^n, \tag{2.13}$$

$$\beta^*_n(a)=\frac{\left(1/a^2;q\right)_n}{2(q;q)_n(1-q^n)}.$$

Corollary 1. *If $|q|$, $|qa|$, $|qa^2|<1$, then*

(2.14)

$$\begin{aligned}
\sum_{n=1}^{\infty}&\frac{a^2q^n}{1-a^2q^n}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}\\
&=\sum_{n=1}^{\infty}(-a)^nq^{n(n+1)/2}-\sum_{n=1}^{\infty}\frac{1-aq^{2n}}{1-a}\frac{(a;q)_n(q;q)_{2n-1}}{(q;q)_n(a^2q;q)_{2n}}(-a)^nq^{n(n+1)/2}\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(1/a^2;q\right)_n a^{2n}q^n}{(q;q)_n(1-q^n)}
\end{aligned}$$

$$-\sum_{n=1}^{\infty}\frac{1-aq^{2n}}{1-a}\frac{(a,a,-aq,1/a;q)_n\,(q^2;q^2)_{n-1}a^nq^n}{(q,q,-1,qa^2;q)_n\,(a^2q^2;q^2)_n}.$$

Proof. Insert the derived pairs, respectively, at (2.11) and (2.13) into (2.1). □

Before proving Theorem 1, we recall the result from Lemma 4 in [9]: if

$$f(a,k,z,q)=\sum_{n=1}^{\infty}\frac{kq^n}{1-kq^n}+\sum_{n=1}^{\infty}\frac{q^na/z}{1-q^na/z}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}-\sum_{n=1}^{\infty}\frac{q^nk/z}{1-q^nk/z}, \tag{2.15}$$

then

$$f(a,k,z,q)-f\left(\frac{1}{a},\frac{1}{k},\frac{1}{z},q\right)=\frac{(a-k)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)} \\ +\frac{z}{k}\frac{(z,q/z,k/a,qa/k,ak/z,qz/ak,q,q;q)_\infty}{(z/k,qk/z,z/a,qa/z,a,q/a,k,q/k;q)_\infty}. \tag{2.16}$$

Proof of Theorem 1. Replace k with b and set $z=-1$ in (2.16), to get (after some simple rearrangements) that

$$f(a,b,-1,q)-f\left(\frac{1}{a},\frac{1}{b},-1,q\right) \\ =\frac{2(a-b)(1+ab)}{(1-a^2)(1-b^2)}-2a\frac{(b/a,qa/b,-ab,-q/ab;q)_\infty(q^2,q^2;q^2)_\infty}{(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}.$$

The result now follows, upon noting that (2.15) and (2.3) imply that

$$f(a,b,-1,q)=\sum_{n=1}^{\infty}\frac{2bq^n}{1-b^2q^{2n}}-\sum_{n=1}^{\infty}\frac{2aq^n}{1-a^2q^{2n}}=f_2(a,q)-f_2(b,q),$$
$$f(1/a,1/b,-1,q)=f_2(1/a,q)-f_2(1/b,q).$$

□

As remarked earlier, any derived WP-Bailey pair (α_n^*,β_n^*) may be used in (1.4), providing the various series involved converge. We note that the left side of (1.4) contains sixteen different infinite series, so for space saving reasons we give two example that uses relatively simple derived pairs. Upon inserting the pair at (2.11) in (1.4) and performing some simple collecting and rearranging of terms, the following identity results.

Corollary 2. *Let a and b be non-zero complex numbers such that a^2q^n, $b^2q^n\neq 1$, for $n\in\mathbb{Z}$. Then*

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+3n+1}}{1-q^{2n+1}}\left[b^{2n+1}-a^{2n+1}+\frac{1}{a^{2n+1}}-\frac{1}{b^{2n+1}}\right] \\ -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(q;q)_{2n-1}}{(q;q)_n}\left[\frac{(1-aq^{2n})(a;q)_n}{(1-a)(qa^2;q)_{2n}}a^{2n}\right. \tag{2.17}$$

$$
\begin{aligned}
&-\frac{(1+aq^{2n})(-a;q)_n}{(1+a)(qa^2;q)_{2n}}a^{2n}-\frac{(1-bq^{2n})(b;q)_n}{(1-b)(qb^2;q)_{2n}}b^{2n}+\frac{(1+bq^{2n})(-b;q)_n}{(1+b)(qb^2;q)_{2n}}b^{2n}\\
&\quad-\frac{(1-q^{2n}/a)(1/a;q)_n}{(1-1/a)(q/a^2;q)_{2n}}a^{-2n}+\frac{(1+q^{2n}/a)(-1/a;q)_n}{(1+1/a)(q/a^2;q)_{2n}}a^{-2n}\\
&\quad+\frac{(1-q^{2n}/b)(1/b;q)_n}{(1-1/b)(q/b^2;q)_{2n}}b^{-2n}-\frac{(1+q^{2n}/b)(-1/b;q)_n}{(1+1/b)(q/b^2;q)_{2n}}b^{-2n}\Bigg]\\
&\quad=\frac{(a-b)(1+ab)}{(1-a^2)(1-b^2)}-a\frac{(b/a,qa/b,-ab,-q/ab;q)_\infty(q^2,q^2;q^2)_\infty}{(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}.
\end{aligned}
$$

One fact about the identity above, and indeed Theorem 1 generally, that is somewhat interesting, is that while the series involving "a" and "b" on the left side of the identity are completely separated and distinct, "a" and "b" are closely bound together on the right side (for example, in the infinite product $(b/a, qa/b, -ab, -q/ab; q)_\infty$). The same comment likewise holds for the next identity.

Corollary 3. *Let a and b be non-zero complex numbers such that aq^n, $bq^n \neq \pm 1$, for $n \in \mathbb{Z}$, and $|q| < \max\{|a^2|, 1/|a^2|, |b^2|, 1/|b^2|\}$. Then*

$$
\begin{aligned}
(2.18)\quad &\sum_{n=1}^{\infty}\Bigg[\left(\frac{(1/a;q)_n}{(qa;q)_n}-\frac{(-1/a;q)_n}{(-qa;q)_n}\right)\frac{a^{2n}q^n}{1-q^n}\\
&\quad-\left(\frac{(a;q)_n}{(q/a;q)_n}-\frac{(-a;q)_n}{(-q/a;q)_n}\right)\frac{q^n}{a^{2n}(1-q^n)}\\
&\quad-\left(\frac{(1/b;q)_n}{(qb;q)_n}-\frac{(-1/b;q)_n}{(-qb;q)_n}\right)\frac{b^{2n}q^n}{1-q^n}\\
&\quad+\left(\frac{(b;q)_n}{(q/b;q)_n}-\frac{(-b;q)_n}{(-q/b;q)_n}\right)\frac{q^n}{b^{2n}(1-q^n)}\Bigg]\\
&=\frac{2(a-b)(1+ab)}{(1-a^2)(1-b^2)}-2a\frac{(b/a,qa/b,-ab,-q/ab;q)_\infty(q^2,q^2;q^2)_\infty}{(a^2,q^2/a^2,b^2,q^2/b^2;q^2)_\infty}.
\end{aligned}
$$

Proof. Insert the derived pair at (2.8) into (1.4). □

3. q-series/products that are representable in terms of certain Lambert Series

Many q-series and q-products have been represented by Ramanujan and others in terms of Lambert series of the types encountered earlier. The various expressions for $f_1(a,q)$ and $f_2(a,q)$ stated previously now permit these q-series and q-products to expressed in several ways as basic hypergeometric series, one way for each derived WP-Bailey pair (or arbitrarily many ways, if a derived pair contains one or more free parameters). We give several examples to illustrate the different ways in which this may be accomplished.

Let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Here we are using the notation for this series employed in [6].

Corollary 4. *If $\rho_1, \rho_2 \neq 0$ and $0 < |q| < 1$, then*

$$\begin{aligned}
(3.1)\quad a(q) &= 1 + 6\sum_{n=1}^{\infty} \frac{(q;q^3)_n q^n}{(q^2;q^3)_n(1-q^{3n})}, \\
&= 1 - 6\sum_{n=1}^{\infty} \frac{(1-q^{6n-1})(1/q,1/q;q^3)_n(q^3;q^3)_{2n-1}q^{2n}}{(1-1/q)(q^3,q^3;q^3)_n(q;q^3)_{2n}}, \\
&= 1 + 6\sum_{n=1}^{\infty} \frac{(\rho_1 q, \rho_2 q, q^2/\rho_1\rho_2;q^3)_n q^n}{(q^2/\rho_1, q^2/\rho_2, \rho_1\rho_2 q;q^3)_n(1-q^{3n})} \\
&\quad - 6\sum_{n=1}^{\infty} \frac{(1-q^{6n-1})(1/q,\rho_1,\rho_2,q/\rho_1\rho_2;q^3)_n(q^3;q^3)_{2n-1}q^{2n}}{(1-1/q)(q^2/\rho_1,q^2/\rho_2,\rho_1\rho_2 q,q^3;q^3)_n(q;q^3)_{2n}}, \\
&= 1 + 6\sum_{n=1}^{\infty} \frac{(-1)^n q^{(3n^2+n)/2}}{1-q^{3n}} \\
&\quad - 6\sum_{n=1}^{\infty} \frac{(1-q^{6n-1})(1/q;q^3)_n(q^3;q^3)_{2n-1}(-1)^n q^{(3n^2-n)/2}}{(1-1/q)(q^3;q^3)_n(q;q^3)_{2n}}.
\end{aligned}$$

Proof. The following result is **Entry 18.2.8** of Ramanujan's Lost Notebook (see [3, page 402]):

$$a(q) = 1 + 6\sum_{n=1}^{\infty} \frac{q^{-2}q^{3n}}{1-q^{-2}q^{3n}} - 6\sum_{n=1}^{\infty} \frac{q^{-1}q^{3n}}{1-q^{-1}q^{3n}}.$$

Thus $a(q) = 1 + 6f_1(1/q, q^3)$, where $f_1(a,q)$ is as defined at (2.2).. The first two equalities follow from the other two representations of $1 + 6f_1(1/q, q^3)$ that derive from the right side of (2.2). The last two equalities follow from substituting the derived pairs at (2.10) and (2.11) into $1 + 6\times$(the left of (2.1)) (with q replaced with q^3 and a replaced with $1/q$). $\square$

Recall that

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Corollary 5. *Let $|\rho_1| > 1$ and $i = \sqrt{-1}$. Then*

(3.2)

$$\begin{aligned}
\psi^2(q^4) &= \frac{i}{2q}\Bigg(\sum_{n=1}^{\infty} \frac{(1+iq^{4n-1})(-i/q,-i/q;q^2)_n(q^2;q^2)_{2n-1}(-iq)^n}{(1+i/q)(q^2,q^2;q^2)_n(-1;q^2)_{2n}} \\
&\quad - \sum_{n=1}^{\infty} \frac{(1-iq^{4n-1})(i/q,i/q;q^2)_n(q^2;q^2)_{2n-1}(iq)^n}{(1-i/q)(q^2,q^2;q^2)_n(-1;q^2)_{2n}}\Bigg), \\
&= \frac{1}{2iq}\Bigg(\sum_{n=1}^{\infty} \frac{(i\rho_1 q;q^2)_n\,(-1)^n}{(-iq/\rho_1;q^2)_n(1-q^{2n})\,\rho_1^n} - \sum_{n=1}^{\infty} \frac{(-i\rho_1 q;q^2)_n\,(-1)^n}{(iq/\rho_1;q^2)_n(1-q^{2n})\,\rho_1^n}
\end{aligned}$$

$$+\sum_{n=1}^{\infty}\frac{(1-iq^{4n-1})(i/q,\rho_1;q^2)_n(q^2;q^2)_{2n-1}(-1)^n}{(1-i/q)(iq/\rho_1,q^2;q^2)_n(-1;q^2)_{2n}\,\rho_1^n}$$
$$-\sum_{n=1}^{\infty}\frac{(1+iq^{4n-1})(-i/q,\rho_1;q^2)_n(q^2;q^2)_{2n-1}(-1)^n}{(1+i/q)(-iq/\rho_1,q^2;q^2)_n(-1;q^2)_{2n}\,\rho_1^n}\Bigg),$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^nq^{4n^2+4n}}{1-q^{4n+2}}+\frac{1}{2iq}\Bigg(\sum_{n=1}^{\infty}\frac{(1-iq^{4n-1})(i/q;q^2)_n(q^2;q^2)_{2n-1}q^{n^2-n}}{(1-i/q)(q^2;q^2)_n(-1;q^2)_{2n}}$$
$$-\sum_{n=1}^{\infty}\frac{(1+iq^{4n-1})(-i/q;q^2)_n(q^2;q^2)_{2n-1}q^{n^2-n}}{(1+i/q)(q^2;q^2)_n(-1;q^2)_{2n}}\Bigg).$$

Proof. By Example (iv) in Section 17 of Chapter 17 of Ramanujan's second notebook (see [4, page 139]),

$$\psi^2(q^2)=\sum_{n=0}^{\infty}\frac{q^n}{1+q^{2n+1}},$$

so that

$$\psi^2(q^4)=\sum_{n=1}^{\infty}\frac{q^{2n-2}}{1+q^{4n-2}}=\frac{1}{2iq}\sum_{n=1}^{\infty}\frac{-2\left(\frac{1}{iq}\right)q^{2n}}{1-\left(\frac{1}{iq}\right)^2q^{4n}}$$
$$=\frac{1}{2iq}f_2\left(\frac{1}{iq},q^2\right)=\frac{1}{2iq}\left(f_1\left(\frac{1}{iq},q^2\right)-f_1\left(\frac{-1}{iq},q^2\right)\right). \tag{3.3}$$

The first equality now follows from (2.2), using the second representations for $f_1(1/iq,q^2)$ and $f_1(-1/iq,q^2)$.

The second equality is a consequence of letting $\rho_2\to\infty$ in the derived pair at (2.10) and substituting the resulting derived pair into the expression for $f_2(1/iq,q^2)/(2iq)$ that follows from (1.3).

The third equality is a consequence of letting $\rho_1\to\infty$ in the second equality. □

For a third example, we recall another identity of Ramanujan (see **Entry 3** (i), Chapter 19, page 223 of [4]):

$$q\psi(q^2)\psi(q^6)=\sum_{n=1}^{\infty}\frac{q^{6n-5}}{1-q^{12n-10}}-\sum_{n=1}^{\infty}\frac{q^{6n-1}}{1-q^{12n-2}}, \tag{3.4}$$

From (2.3),

$$q\psi(q^2)\psi(q^6)=\frac{1}{2}\left(f_2(q^{-1},q^6)-f_2(q^{-5},q^6)\right)$$
$$=\frac{1}{2}\left(f_1(q^{-1},q^6)-f_1(-q^{-1},q^6)-f_1(q^{-5},q^6)+f_1(-q^{-5},q^6)\right)$$

Thus any derived pair $(\alpha_n^*(a,q),\beta_n^*(a,q))$ inserted in Lemma 1, with q replaced with q^6 and a taking the values $q^{-1},-q^{-1},q^{-5},-q^{-5}$ will give an expression for $q\psi(q^2)\psi(q^6)$ containing 8 series. However, for simplicity, we use the pair at (2.6) (so $\beta_n^*(a)=0$, reducing the 8 series to 4) to get the following identity.

Corollary 6.

$$\begin{aligned}2q\psi(q^2)\psi(q^6) &= \sum_{n=1}^{\infty}\frac{1+q^{12n-1}}{1+1/q}\frac{(-1/q,-1/q;q^6)_n(q^6;q^6)_{2n-1}(-q)^{5n}}{(q^6,q^6;q^6)_n(q^4;q^6)_{2n}}\\ &\quad-\sum_{n=1}^{\infty}\frac{1-q^{12n-1}}{1-1/q}\frac{(1/q,1/q;q^6)_n(q^6;q^6)_{2n-1}q^{5n}}{(q^6,q^6;q^6)_n(q^4;q^6)_{2n}}\\ &\quad+\sum_{n=1}^{\infty}\frac{1-q^{12n-5}}{1-1/q^5}\frac{(1/q^5,1/q^5;q^6)_n(q^6;q^6)_{2n-1}q^{n}}{(q^6,q^6;q^6)_n(1/q^4;q^6)_{2n}}\\ &\quad-\sum_{n=1}^{\infty}\frac{1+q^{12n-5}}{1+1/q^5}\frac{(-1/q^5,-1/q^5;q^6)_n(q^6;q^6)_{2n-1}(-q)^{n}}{(q^6,q^6;q^6)_n(1/q^4;q^6)_{2n}}.\end{aligned}$$

There are a number of other identities, for example **Entry 34 (p.284)** in chapter 36 of Ramanujan's notebooks (see [5, page 374]),

$$q\frac{\psi^3(q^3)}{\psi(q)}=\sum_{n=1}^{\infty}\frac{q^{3n-2}}{1-q^{6n-4}}-\sum_{n=1}^{\infty}\frac{q^{3n-1}}{1-q^{6n-2}}, \tag{3.5}$$

where theta functions are expressed in terms of certain Lambert series, which may be treated similarly to derive results like those in this section.

4. Concluding Remarks

In the present paper and its companion [9] we considered limiting cases of the two WP-Bailey chains described by Andrews in [1]. There are a number of other WP-Bailey chains described in the literature (see the papers of Warnaar [14], Liu and Ma [8] and Mc Laughlin and Zimmer [10]), and it may be that a similar analysis of some of these chains may also have interesting consequences.

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