SOME NEW TRANSFORMATIONS FOR BAILEY PAIRS AND WP-BAILEY PAIRS

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ABSTRACT. We derive several new transformations relating WP-Bailey pairs. We also consider the corresponding relations relating standard Bailey pairs, and as a consequence, derive some quite general expansions for products of theta functions which can also be expressed as certain types of Lambert series.

1. INTRODUCTION

Andrews [1], building on previous work of Bressoud [8] and Singh [14], defined a *WP-Bailey pair* to be a pair of sequences $(\alpha_n(a,k,q),\ \beta_n(a,k,q))$ (if the context is clear, we occasionally suppress the dependence on some or all of a, k and q) satisfying $\alpha_0(a,k,q) = \beta_0(a,k,q)$ and

$$\beta_n(a,k,q) = \sum_{j=0}^{n} \frac{(k/a;q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(aq;q)_{n+j}} \alpha_j(a,k,q). \tag{1.1}$$

If $k=0$, then the pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a *Bailey pair with respect to* a.

In the same paper Andrews showed that if the pair $(\alpha_n(a,k),\ \beta_n(a,k))$ satisfies (1.1), then so does $(\alpha'_n(a,k),\ \beta'_n(a,k))$ where

$$\begin{aligned} \alpha'_n(a,k) &= \frac{(y,z;q)_n}{(aq/y,aq/z;q)_n}\left(\frac{k}{c}\right)^n \alpha_n(a,c), \\ \beta'_n(a,k) &= \frac{(ky/a,kz/a;q)_n}{(aq/y,aq/z;q)_n} \\ &\quad\times \sum_{j=0}^{n} \frac{(1-cq^{2j})(y,z;q)_j(k/c;q)_{n-j}(k;q)_{n+j}}{(1-c)(ky/a,kz/a;q)_n(q;q)_{n-j}(qc;q)_{n+j}}\left(\frac{k}{c}\right)^j \beta_j(a,c), \end{aligned} \tag{1.2}$$

with $c = kyz/aq$. Andrews [1] also described a second method for deriving new WP-Bailey pairs from existing pairs, but this second method will not concern us in the present paper.

These two constructions allow a "tree" of WP-Bailey pairs to be generated from a single WP-Bailey pair. The implications of these two branches

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were further investigated by Andrews and Berkovich in [2]. Spiridonov [16] derived an elliptic generalization of Andrews first WP-Bailey chain. Four additional branches were added to the WP-Bailey tree by Warnaar [18], two of which had generalizations to the elliptic level. More recently, Liu and Ma [10] introduced the idea of a general WP-Bailey chain, and added one new branch to the WP-Bailey tree. In [12], the authors added three new WP-Bailey chains.

It is not difficult to show (see Corollary 1 in [13], for example) that the WP-Bailey chain at (1.2) implies that if (α_n, β_n) satisfy (1.1), then subject to suitable convergence conditions,

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z; q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z; q)_n} \left(\frac{qa}{yz}\right)^n \beta_n = \frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(qa/y, qa/z; q)_n} \left(\frac{qa}{yz}\right)^n \alpha_n. \tag{1.3}$$

In the present paper we prove some new relations for WP-Bailey pairs. These include the following.

Theorem 1. *If $(\alpha_n(a,k), \beta_n(a,k))$ is a WP-Bailey pair, then subject to suitable convergence conditions,*

$$\sum_{n=1}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, z; q)_n (q;q)_{n-1}}{\left(\sqrt{k}, -\sqrt{k}, qk, \frac{qk}{z}; q\right)_n} \left(\frac{qa}{z}\right)^n \beta_n(a,k) - \sum_{n=1}^{\infty} \frac{\left(q\sqrt{\frac{1}{k}}, -q\sqrt{\frac{1}{k}}, \frac{1}{z}; q\right)_n (q;q)_{n-1}}{\left(\sqrt{\frac{1}{k}}, -\sqrt{\frac{1}{k}}, \frac{q}{k}, \frac{qz}{k}; q\right)_n} \left(\frac{qz}{a}\right)^n \beta_n\left(\frac{1}{a}, \frac{1}{k}\right) - \sum_{n=1}^{\infty} \frac{(z;q)_n (q;q)_{n-1}}{\left(qa, \frac{qa}{z}; q\right)_n} \left(\frac{qa}{z}\right)^n \alpha_n(a,k) + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{z}; q\right)_n (q;q)_{n-1}}{\left(\frac{q}{a}, \frac{qz}{a}; q\right)_n} \left(\frac{qz}{a}\right)^n \alpha_n\left(\frac{1}{a}, \frac{1}{k}\right) = \frac{(a-k)\left(1-\frac{1}{z}\right)\left(1-\frac{ak}{z}\right)}{(1-a)(1-k)\left(1-\frac{a}{z}\right)\left(1-\frac{k}{z}\right)} + \frac{z}{k} \frac{\left(z, \frac{q}{z}, \frac{k}{a}, \frac{qa}{k}, \frac{ak}{z}, \frac{qz}{ak}, q, q; q\right)_\infty}{\left(\frac{z}{k}, \frac{qk}{z}, \frac{z}{a}, \frac{qa}{z}, a, \frac{q}{a}, k, \frac{q}{k}; q\right)_\infty}. \tag{1.4}$$

Theorem 2. *If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ is a WP-Bailey pair, then subject to suitable convergence conditions,*

$$\sum_{n=1}^{\infty} \frac{(1-kq^{2n})(z;q)_n (q;q)_{n-1}}{(1-k)(qk, qk/z; q)_n} \left(\frac{qa}{z}\right)^n \beta_n(a,k,q) + \sum_{n=1}^{\infty} \frac{(1+kq^{2n})(z;q)_n (q;q)_{n-1}}{(1+k)(-qk, -qk/z; q)_n} \left(\frac{-qa}{z}\right)^n \beta_n(-a,-k,q) - 2\sum_{n=1}^{\infty} \frac{(1-k^2q^{4n})(z^2;q^2)_n (q^2;q^2)_{n-1}}{(1-k^2)(q^2k^2, q^2k^2/z^2; q^2)_n} \left(\frac{q^2a^2}{z^2}\right)^n \beta_n(a^2,k^2,q^2) \tag{1.5}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{(z;q)_n (q;q)_{n-1}}{(qa, qa/z;q)_n} \left(\frac{qa}{z}\right)^n \alpha_n(a,k,q) \\
&+ \sum_{n=1}^{\infty} \frac{(z;q)_n (q;q)_{n-1}}{(-qa, -qa/z;q)_n} \left(\frac{-qa}{z}\right)^n \alpha_n(-a,-k,q) \\
&\quad - 2\sum_{n=1}^{\infty} \frac{(z^2;q^2)_n (q^2;q^2)_{n-1}}{(q^2a^2, q^2a^2/z^2;q^2)_n} \left(\frac{q^2a^2}{z^2}\right)^n \alpha_n(a^2,k^2,q^2).
\end{aligned}
$$

We find some similar relations for standard Bailey pairs and derive some interesting consequences. For example, if $d \neq q^{3n\pm 1}$, then

$$
\begin{aligned}
q\frac{\psi^3(q^3)}{\psi(q)} &= \sum_{n=1}^{\infty} \frac{(q^6;q^6)_{n-1}(q^2/d;q^6)_n(-q^2)^n}{(q^2;q^6)_n(q^2/d,q^3;q^3)_n} \\
&\quad - \sum_{n=1}^{\infty} \frac{(q^6;q^6)_{n-1}(q/d;q^6)_n(-q)^n}{(q;q^6)_n(q/d,q^3;q^3)_n} \\
&+ \sum_{n=1}^{\infty} \frac{(1-q^{12n-2})(q^6;q^6)_{2n-1}(1/q^2,d;q^6)_n q^{6n^2}}{(1-1/q^2)(q^2;q^6)_{2n}(q^4/d,q^6;q^6)_n d^n} \\
&\quad - \sum_{n=1}^{\infty} \frac{(1-q^{12n-1})(q^6;q^6)_{2n-1}(1/q,d;q^6)_n q^{6n^2+3n}}{(1-1/q)(q^4;q^6)_{2n}(q^5/d,q^6;q^6)_n d^n}.
\end{aligned}
$$

Here $\psi(q)$ is Ramanujan's theta function. We show that similar results hold for many other theta products.

We use the standard notations:

$$
\begin{aligned}
(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}), \\
(a_1,a_2,\dots,a_j;q)_n &:= (a_1;q)_n(a_2;q)_n\cdots(a_j;q)_n, \\
(a;q)_\infty &:= (1-a)(1-aq)(1-aq^2)\cdots, \text{ and} \\
(a_1,a_2,\dots,a_j;q)_\infty &:= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_j;q)_\infty,
\end{aligned}
$$

We will make use Bailey's ${}_6\psi_6$ summation formula [17].

$$
\begin{aligned}
&\frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a;q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde;q)_\infty} \qquad (1.6)\\
&= \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(b,c,d,e;q)_n}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_n}\left(\frac{qa^2}{bcde}\right)^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(1-aq^{2n})(b,c,d,e;q)_n}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_n}\left(\frac{qa^2}{bcde}\right)^n \\
&\quad + \sum_{n=1}^{\infty} \frac{(1-1/aq^{2n})(b/a,c/a,d/a,e/a;q)_n}{(1-1/a)(q/b,q/c,q/d,q/e;q)_n}\left(\frac{qa^2}{bcde}\right)^n,
\end{aligned}
$$

where the second equality follows from the definition

$$(z;q)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{z^n (q/z;q)_n}.$$

We also recall Jackson's summation formula for a very-well-poised ${}_6\phi_5$ series [9, p. 356, Eq. (II. 20)] (which follows upon setting $e = a$ in (1.6)):

(1.7)
$$\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d;q)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d};q\right)_n} \left(\frac{aq}{bcd}\right)^n = \frac{(aq, aq/bc, aq/bd, aq/cd;q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd;q)_\infty}.$$

Finally, we make use of the q-Binomial Theorem,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.8}$$

Unless stated otherwise, we assume $|q| < 1$.

2. Proofs of the Main Identities

The next transformation follows easily from the identity at (1.3).

Lemma 1. *If (α_n, β_n) is a WP-Bailey pair, then subject to suitable convergence conditions,*

(2.1)
$$\sum_{n=1}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, z;q)_n (q;q)_{n-1}}{(\sqrt{k}, -\sqrt{k}, qk, qk/z;q)_n} \left(\frac{qa}{z}\right)^n \beta_n - \sum_{n=1}^{\infty} \frac{(z;q)_n (q;q)_{n-1}}{(qa, qa/z;q)_n} \left(\frac{qa}{z}\right)^n \alpha_n$$
$$= \sum_{n=1}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, k, z, k/a;q)_n}{(\sqrt{k}, -\sqrt{k}, qk, qk/z, qa;q)_n (1-q^n)} \left(\frac{qa}{z}\right)^n.$$

Proof. Rewrite (1.3) as

$$\sum_{n=1}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, z;q)_n (yq;q)_{n-1}}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z;q)_n} \left(\frac{qa}{yz}\right)^n \beta_n$$
$$- \frac{(qk, qk/yz, qa/y, qa/z;q)_\infty}{(qk/y, qk/z, qa, qa/yz;q)_\infty} \sum_{n=1}^{\infty} \frac{(z;q)_n (yq;q)_{n-1}}{(qa/y, qa/z;q)_n} \left(\frac{qa}{yz}\right)^n \alpha_n =$$
$$\frac{1}{1-y} \left(\frac{(qk, qk/yz, qa/y, qa/z;q)_\infty}{(qk/y, qk/z, qa, qa/yz;q)_\infty} - 1 \right).$$

From (1.7) it can be seen that

(2.2)
$$\frac{(qk, qk/yz, qa/y, qa/z;q)_\infty}{(qk/y, qk/z, qa, qa/yz;q)_\infty} = \sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, k, y, z, k/a;q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, qa, q;q)_n} \left(\frac{qa}{yz}\right)^n,$$

and the result now follows upon letting $y \to 1$. □

For later use we note that the first series on the right side of (2.1) has the following properties. We define

$$f(a,k,z,q) := \sum_{n=1}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, k, z, k/a; q)_n}{(\sqrt{k}, -\sqrt{k}, qk, qk/z, qa; q)_n (1-q^n)} \left(\frac{qa}{z}\right)^n. \tag{2.3}$$

Lemma 2. *If $|qa|, |qk| < |z|$ and if none of the denominators vanish, then*

$$f(a,k,z,q) = -f(k,a,z,q). \tag{2.4}$$

Proof. This immediate upon writing

$$\begin{aligned}&\frac{1}{1-y}\left(\frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty} - 1\right)\\ &= \frac{1}{1-y}\left(1 - \frac{(qk/y, qk/z, qa, qa/yz; q)_\infty}{(qk, qk/yz, qa/y, qa/z; q)_\infty}\right)\frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty},\end{aligned}$$

using (1.7) on the infinite product inside the brackets, and again letting $y \to 1$. □

We remark in passing that the expansion at (2.2) and the similar expansion of the reciprocal of this product imply that if

$$g(a,k,y,z,q) := \sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, k, y, z, k/a; q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, qa, q; q)_n} \left(\frac{qa}{yz}\right)^n,$$

then

$$g(a,k,y,z,q) = \frac{1}{g(k,a,y,z,q)}. \tag{2.5}$$

We next express $f(a,k,z,q)$ as a sum of Lambert series.

Lemma 3. *If $|qa| < |z|$ and if none of the denominators vanish, then*

$$f(a,k,z,q) = \sum_{n=1}^{\infty} \frac{kq^n}{1-kq^n} + \sum_{n=1}^{\infty} \frac{q^n a/z}{1-q^n a/z} - \sum_{n=1}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{q^n k/z}{1-q^n k/z}. \tag{2.6}$$

Proof. If we define

$$G(y) := \frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty}$$

we see that

$$\begin{aligned}f(a,k,z,q) &= \lim_{y\to 1} \frac{1}{1-y}\left(\frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty} - 1\right)\\ &= \lim_{y\to 1} \frac{G(y) - G(1)}{1-y} = -G'(1).\end{aligned}$$

That $-G'(1)$ equals the right side of (2.6) follows easily. □

Lemma 4. *If $|qa| < |z|$ and if none of the denominators vanish, then*

$$f(a,k,z,q) + f(-a,-k,z,q) = 2f(a^2,k^2,z^2,q^2). \tag{2.7}$$

Proof. This is an immediate consequence of Lemma 3, upon employing the elementary identity

$$\frac{x}{1-x}+\frac{(-x)}{1-(-x)}=\frac{2x^2}{1-x^2}.$$

□

Remark: By somewhat similar reasoning, one can show that if $m \geq 2$ is a positive integer and ω is a primitive m-root of unity, then

$$\sum_{j=0}^{m-1} f(a\omega^j, k\omega^j, z, q) = mf(a^m, k^m, z^m, q^m).$$

Proof of Theorem 2. This follows immediately from Lemma 4. □

One could easily insert specific WP-Bailey pairs in (1.5) to provide explicit identities, but we leave that to the reader. We also note that letting $k \to 0$ in Corollary 2 gives a result for standard Bailey pairs.

Corollary 1. *If $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with respect to a, then subject to suitable convergence conditions,*

(2.8)
$$\sum_{n=1}^{\infty}(z;q)_n(q;q)_{n-1}\left(\frac{qa}{z}\right)^n\beta_n(a,q)+\sum_{n=1}^{\infty}(z;q)_n(q;q)_{n-1}\left(\frac{-qa}{z}\right)^n\beta_n(-a,q)$$
$$-2\sum_{n=1}^{\infty}(z^2;q^2)_n(q^2;q^2)_{n-1}\left(\frac{q^2a^2}{z^2}\right)^n\beta_n(a^2,q^2)$$
$$=\sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{(qa,qa/z;q)_n}\left(\frac{qa}{z}\right)^n\alpha_n(a,q)$$
$$+\sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{(-qa,-qa/z;q)_n}\left(\frac{-qa}{z}\right)^n\alpha_n(-a,q)$$
$$-2\sum_{n=1}^{\infty}\frac{(z^2;q^2)_n(q^2;q^2)_{n-1}}{(q^2a^2,q^2a^2/z^2;q^2)_n}\left(\frac{q^2a^2}{z^2}\right)^n\alpha_n(a^2,q^2).$$

Once again we leave to the reader to produce particular identities, by inserting specific Bailey pairs.

Lemma 5. *If $|qa| < |z| < |a/q|$ and if none of the denominators vanish, then*

$$f(a,k,z,q)-f\left(\frac{1}{a},\frac{1}{k},\frac{1}{z},q\right)=\frac{(a-k)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)}$$
$$+\frac{z}{k}\frac{(z,q/z,k/a,qa/k,ak/z,qz/ak,q,q;q)_\infty}{(z/k,qk/z,z/a,qa/z,a,q/a,k,q/k;q)_\infty}. \tag{2.9}$$

Proof. One can check (preferably with a computer algebra system) that

$$\frac{kq^n}{1-kq^n}+\frac{q^n a/z}{1-q^n a/z}-\frac{aq^n}{1-aq^n}-\frac{q^n k/z}{1-q^n k/z}$$
$$=\frac{(a-k)q^n\left(1-\frac{akq^{2n}}{z}\right)(1-z)}{(1-aq^n)(1-kq^n)\left(1-\frac{aq^n}{z}\right)\left(1-\frac{kq^n}{z}\right)z}$$
$$=\frac{(k-a)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)}\frac{\left(1-\frac{akq^{2n}}{z}\right)(a,k,a/z,k/z;q)_n q^n}{(1-ak/z)(aq,kq,aq/z,kq/z;q)_n},$$

so that

$$f(a,k,z,q)$$
$$=\frac{(k-a)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)}\sum_{n=1}^{\infty}\frac{\left(1-\frac{akq^{2n}}{z}\right)(a,k,a/z,k/z;q)_n q^n}{(1-ak/z)(aq,kq,aq/z,kq/z;q)_n}$$

Similarly, it can be shown that

$$f\left(\frac{1}{a},\frac{1}{k},\frac{1}{z},q\right)=-\frac{(k-a)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)}$$
$$\times\sum_{n=1}^{\infty}\frac{\left(1-\frac{zq^{2n}}{ak}\right)(1/a,1/k,z/a,z/k;q)_n q^n}{(1-z/ak)(q/a,q/k,qz/a,qz/k;q)_n}.$$

The result now follows, since

$$f(a,k,z,q)-f\left(\frac{1}{a},\frac{1}{k},\frac{1}{z},q\right)+\frac{(k-a)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)}$$

is easily seen to equal a constant times a bilateral series which is summable by Bailey's ${}_6\psi_6$ summation formula at (1.6), with some further easy manipulations giving the final result. □

Remark: The proof that the sum of Lambert series above combine to give the stated infinite product was first given by Andrews, Lewis and Liu in [4] (using a different labeling for the parameters) in a different context, so they did not have our reciprocity result for the basic hypergeometric series $f(a,k,z,q)$.

Note that substituting the expression for $f(a,k,z,q)$ from (2.3) into (2.9) leads to the identity

$$(2.10)\quad \sum_{n=1}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k},k,z,\frac{k}{a};q)_n(q;q)_{n-1}}{\left(\sqrt{k},-\sqrt{k},kq,qa,\frac{qk}{z},q;q\right)_n}\left(\frac{qa}{z}\right)^n$$
$$-\sum_{n=1}^{\infty}\frac{\left(\frac{q}{\sqrt{k}},\frac{-q}{\sqrt{k}},\frac{1}{k},\frac{1}{z},\frac{a}{k};q\right)_n(q;q)_{n-1}}{\left(\frac{1}{\sqrt{k}},\frac{-1}{\sqrt{k}},\frac{q}{k},\frac{qz}{k},\frac{q}{a},q;q\right)_n}\left(\frac{qz}{a}\right)^n$$

$$= \frac{(a-k)\left(1-\frac{1}{z}\right)\left(1-\frac{ak}{z}\right)}{(1-a)(1-k)\left(1-\frac{a}{z}\right)\left(1-\frac{k}{z}\right)} + \frac{z}{k}\frac{\left(z,\frac{q}{z},\frac{k}{a},\frac{qa}{k},\frac{ak}{z},\frac{qz}{ak},q,q;q\right)_\infty}{\left(\frac{z}{k},\frac{qk}{z},\frac{z}{a},\frac{qa}{z},a,\frac{q}{a},k,\frac{q}{k};q\right)_\infty},$$

an identity which does not appear to follow directly from Bailey's formula at (1.6). We are now ready to prove Theorem 1.

Proof of Theorem 1. Take the identity at (2.1) and replace a with $1/a$, k with $1/k$ and z with $1/z$. Then subtract the resulting identity from the original identity, use (2.9) to replace two of the sums on the right, and (1.4) follows. □

Any WP-Bailey that is inserted into (1.4) will lead to a summation formula for basic hypergeometric series. We give two example as illustrations.

Corollary 2.

$$\begin{aligned}(2.11)\quad &\sum_{n=1}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k},z,k,\frac{k\rho_1}{a},\frac{k\rho_2}{a},\frac{aq}{\rho_1\rho_2};q)_n(q;q)_{n-1}}{\left(\sqrt{k},-\sqrt{k},qk,\frac{qk}{z},\frac{aq}{\rho_1},\frac{aq}{\rho_2},\frac{k\rho_1\rho_2}{a},q;q\right)_n}\left(\frac{qa}{z}\right)^n\\
&-\sum_{n=1}^{\infty}\frac{\left(\frac{q}{\sqrt{k}},\frac{-q}{\sqrt{k}},\frac{1}{z},\frac{1}{k},\frac{a\rho_1}{k},\frac{a\rho_2}{k},\frac{q}{a\rho_1\rho_2};q\right)_n(q;q)_{n-1}}{\left(\frac{1}{\sqrt{k}},\frac{-1}{\sqrt{k}},\frac{q}{k},\frac{qz}{k},\frac{q}{a\rho_1},\frac{q}{a\rho_2},\frac{a\rho_1\rho_2}{k},q;q\right)_n}\left(\frac{qz}{a}\right)^n\\
&-\sum_{n=1}^{\infty}\frac{(q\sqrt{a},-q\sqrt{a},a,\rho_1,\rho_2,\frac{a^2q}{k\rho_1\rho_2},z;q)_n(q;q)_{n-1}}{\left(\sqrt{a},-\sqrt{a},\frac{aq}{\rho_1},\frac{aq}{\rho_2},\frac{k\rho_1\rho_2}{a},qa,\frac{qa}{z},q;q\right)_n}\left(\frac{qk}{z}\right)^n\\
&+\sum_{n=1}^{\infty}\frac{\left(\frac{q}{\sqrt{a}},\frac{-q}{\sqrt{a}},\frac{1}{a},\rho_1,\rho_2,\frac{kq}{a^2\rho_1\rho_2},\frac{1}{z};q\right)_n(q;q)_{n-1}}{\left(\frac{1}{\sqrt{a}},\frac{-1}{\sqrt{a}},\frac{q}{a\rho_1},\frac{q}{a\rho_2},\frac{a\rho_1\rho_2}{k},\frac{q}{a},\frac{qz}{a},q;q\right)_n}\left(\frac{qz}{k}\right)^n\\
&= \frac{(a-k)\left(1-\frac{1}{z}\right)\left(1-\frac{ak}{z}\right)}{(1-a)(1-k)\left(1-\frac{a}{z}\right)\left(1-\frac{k}{z}\right)} + \frac{z}{k}\frac{\left(z,\frac{q}{z},\frac{k}{a},\frac{qa}{k},\frac{ak}{z},\frac{qz}{ak},q,q;q\right)_\infty}{\left(\frac{z}{k},\frac{qk}{z},\frac{z}{a},\frac{qa}{z},a,\frac{q}{a},k,\frac{q}{k};q\right)_\infty}.\end{aligned}$$

Proof. Insert Singh's WP-Bailey pair [14],

$$\begin{aligned}\alpha_n(a,k) &= \frac{(q\sqrt{a},-q\sqrt{a},a,\rho_1,\rho_2,a^2q/k\rho_1\rho_2;q)_n}{(\sqrt{a},-\sqrt{a},q,aq/\rho_1,aq/\rho_2,k\rho_1\rho_2/a;q)_n}\left(\frac{k}{a}\right)^n,\\
\beta_n(a,k) &= \frac{(k\rho_1/a,k\rho_2/a,k,aq/\rho_1\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2,k\rho_1\rho_2/a,q;q)_n},\end{aligned}$$

into (1.4). □

Corollary 3.

$$(2.12)\quad \sum_{n=1}^{\infty}\frac{\left(q\sqrt{k},-q\sqrt{k},z,\frac{k^2}{qa^2};q\right)_n(q;q)_{n-1}}{\left(\sqrt{k},-\sqrt{k},qk,\frac{qk}{z},q;q\right)_n}\left(\frac{qa}{z}\right)^n$$

$$
\begin{aligned}
&-\sum_{n=1}^{\infty}\frac{\left(\frac{q}{\sqrt{k}},-\frac{q}{\sqrt{k}},\frac{1}{z},\frac{a^2}{qk^2};q\right)_n (q;q)_{n-1}}{\left(\frac{1}{\sqrt{k}},-\frac{1}{\sqrt{k}},\frac{q}{k},\frac{qz}{k},q;q\right)_n}\left(\frac{qz}{a}\right)^n\\
&-\sum_{n=1}^{\infty}\frac{(q\sqrt{a},-q\sqrt{a},a,z,\frac{k}{aq};q)_n\left(\frac{qa^2}{k},q\right)_{2n}(q;q)_{n-1}}{\left(\sqrt{a},-\sqrt{a},qa,\frac{qa}{z},\frac{a^2q^2}{k},q;q\right)_n (k;q)_{2n}}\left(\frac{qk}{z}\right)^n\\
&+\sum_{n=1}^{\infty}\frac{\left(\frac{q}{\sqrt{a}},\frac{-q}{\sqrt{a}},\frac{1}{a},\frac{1}{z},\frac{a}{kq};q\right)_n\left(\frac{qk}{a^2},q\right)_{2n}(q;q)_{n-1}}{\left(\frac{1}{\sqrt{a}},\frac{-1}{\sqrt{a}},\frac{q}{a},\frac{qz}{a},\frac{kq^2}{a^2},q;q\right)_n\left(\frac{1}{k},q\right)_{2n}}\left(\frac{qz}{k}\right)^n\\
&=\frac{(a-k)\left(1-\frac{1}{z}\right)\left(1-\frac{ak}{z}\right)}{(1-a)(1-k)\left(1-\frac{a}{z}\right)\left(1-\frac{k}{z}\right)}+\frac{z}{k}\frac{\left(z,\frac{q}{z},\frac{k}{a},\frac{qa}{k},\frac{ak}{z},\frac{qz}{ak},q,q;q\right)_\infty}{\left(\frac{z}{k},\frac{qk}{z},\frac{z}{a},\frac{qa}{z},a,\frac{q}{a},k,\frac{q}{k};q\right)_\infty}.
\end{aligned}
$$

Proof. Insert the WP-Bailey pair

$$
\begin{aligned}
\alpha_n(a,k)&=\frac{(q\sqrt{a},-q\sqrt{a},a,k/aq;q)_n}{(\sqrt{a},-\sqrt{a},q,a^2q^2/k;q)_n}\frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\left(\frac{k}{a}\right)^n,\\
\beta_n(a,k)&=\frac{(k^2/qa^2;q)_n}{(q;q)_n},
\end{aligned}
$$

into (1.4). □

3. Applications to Bailey Pairs

If we let $k\to 0$ in Lemma 1 and Lemmas 2 and 3, we get the following result.

Theorem 3. *If (α_n,β_n) is a Bailey pair with respect to a, then subject to suitable convergence conditions,*

(3.1)
$$
\sum_{n=1}^{\infty}(z;q)_n(q;q)_{n-1}\left(\frac{qa}{z}\right)^n\beta_n-\sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{(qa,qa/z;q)_n}\left(\frac{qa}{z}\right)^n\alpha_n=f_1(a,z,q),
$$

where

$$
\begin{aligned}
f_1(a,z,q)&=-\sum_{n=1}^{\infty}\frac{(q\sqrt{a},-q\sqrt{a},a,z;q)_n q^{n(n+1)/2}}{(\sqrt{a},-\sqrt{a},qa,qa/z;q)_n(1-q^n)}\left(\frac{-a}{z}\right)^n\\
&=\sum_{n=1}^{\infty}\frac{(z;q)_n}{(qa;q)_n(1-q^n)}\left(\frac{qa}{z}\right)^n\\
&=\sum_{n=1}^{\infty}\frac{aq^n/z}{1-aq^n/z}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}.
\end{aligned}
$$

Note that the first two representations for $f_1(a,z,q)$ follow from (3.1), upon inserting, respectively, the "unit" Bailey pair

$$\alpha_n(a,q) = \frac{(q\sqrt{a}, -q\sqrt{a}, a; q)_n}{(\sqrt{a}, -\sqrt{a}, q; q)_n} (-1)^n q^{n(n-1)/2},$$
$$\beta_n(a,q) = \begin{cases} 1 & n=0, \\ 0, & n>1, \end{cases}$$

and the "trivial" Bailey pair

$$\alpha_n(a,q) = \begin{cases} 1 & n=0, \\ 0, & n>1, \end{cases}$$
$$\beta_n(a,q) = \frac{1}{(aq, q; q)_n}.$$

However, here and subsequently, we prefer to write these representations explicitly. Upon letting $z \to \infty$ the following identity results.

Corollary 4. *If* (α_n, β_n) *is a Bailey pair with respect to* a*, then subject to suitable convergence conditions,*

(3.2)
$$\sum_{n=1}^{\infty} (q;q)_{n-1} (-a)^n q^{n(n+1)/2} \beta_n - \sum_{n=1}^{\infty} \frac{(q;q)_{n-1} (-a)^n q^{n(n+1)/2}}{(qa;q)_n} \alpha_n = f_2(a,q),$$

where

$$f_2(a,q) = -\sum_{n=1}^{\infty} \frac{(1-aq^{2n})q^{n^2} a^n}{(1-aq^n)(1-q^n)} \tag{3.3}$$
$$= \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2} (-a)^n}{(qa;q)_n (1-q^n)}$$
$$= -\sum_{n=1}^{\infty} \frac{aq^n}{1-aq^n}.$$

As is well known, many theta products/series can be represented as sums of Lambert series of the type immediately above. The other representations of $f_2(a,q)$ now let these theta functions be represented in two different ways as basic hypergeometric series.

Let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Here we are using the notation for this series employed in [7].

Corollary 5.

$$a(q) = 1 - 6\sum_{n=1}^{\infty} \frac{(-1)^n q^{(3n^2-n)/2}}{(q;q^3)_n (1-q^{3n})} + 6\sum_{n=1}^{\infty} \frac{(-1)^n q^{(3n^2+n)/2}}{(q^2;q^3)_n (1-q^{3n})}, \tag{3.4}$$

$$= 1 + 6\sum_{n=1}^{\infty} \frac{q^{3n^2-2n}(1-q^{6n-2})}{(1-q^{3n-2})(1-q^{3n})} - 6\sum_{n=1}^{\infty} \frac{q^{3n^2-n}(1-q^{6n-1})}{(1-q^{3n-1})(1-q^{3n})}.$$

Proof. The following result is **Entry 18.2.8** of Ramanujan's Lost Notebook (see [3, page 402]):

$$a(q) = 1 + 6\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^n} \tag{3.5}$$
$$= 1 + 6\sum_{n=1}^{\infty} \frac{q^{-2}q^{3n}}{1-q^{-2}q^{3n}} - 6\sum_{n=1}^{\infty} \frac{q^{-1}q^{3n}}{1-q^{-1}q^{3n}}.$$

Use (3.3) (with q replaced with q^3 and $a = q^{-1}$ and $a = q^{-2}$, respectively) to replace each of the Lambert series with, in turn, each of the other two representations of $f_2(a, q^3)$, and the result follows. □

Remark: It is clear that a quite general statement concerning $a(q)$ may be deduced from (3.4) by a similar argument. Indeed, if $(\alpha_n(a,q),\ \beta_n(a,q))$ is *any* Bailey pair in which a is a free parameter, then

$$\begin{aligned} a(q) = 1 &+ 6\sum_{n=1}^{\infty} (q^3;q^3)_{n-1}\,(-1)^n\,q^{(3n^2+n)/2}\beta_n(q^{-1},q^3) \\ &- 6\sum_{n=1}^{\infty} (q^3;q^3)_{n-1}\,(-1)^n\,q^{(3n^2-n)/2}\beta_n(q^{-2},q^3) \\ &- 6\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}\,(-1)^n\,q^{(3n^2+n)/2}\alpha_n(q^{-1},q^3)}{(q^2;q^3)_n} \\ &+ 6\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}\,(-1)^n\,q^{(3n^2-n)/2}\alpha_n(q^{-2},q^3)}{(q;q^3)_n}. \end{aligned} \tag{3.6}$$

As an example, if we insert the Bailey pair of Slater [15, Equation (4.1), page 469],

$$\alpha_n(a,q) = \frac{(1-aq^{2n})(a,c,d;q)_n}{(1-a)(aq/c,aq/d,q;q)_n}\left(\frac{-a}{cd}\right)^n q^{(n^2+n)/2},$$
$$\beta_n(a,q) = \frac{(aq/cd;q)_n}{(aq/c,aq/d,q;q)_n},$$

in (3.6), we get, for any values for c and d that do not make any denominator vanish, that

$$\begin{aligned} a(q) = 1 &+ 6\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}(q^2/cd;q^3)_n\,(-1)^n\,q^{(3n^2+n)/2}}{(q^2/c,q^2/d,q^3;q^3)_n} \\ &- 6\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}(q/cd;q^3)_n\,(-1)^n\,q^{(3n^2-n)/2}}{(q/c,q/d,q^3;q^3)_n} \end{aligned} \tag{3.7}$$

$$-6\sum_{n=1}^{\infty}\frac{(1-q^{6n-1})(q^3;q^3)_{n-1}(1/q,c,d;q^3)_n q^{3n^2+n}}{(1-1/q)(q^2/c,q^2/d,q^2,q^3;q^3)_n c^n d^n}$$
$$+6\sum_{n=1}^{\infty}\frac{(1-q^{6n-2})(q^3;q^3)_{n-1}(1/q^2,c,d;q^3)_n q^{3n^2-n}}{(1-1/q^2)(q/c,q/d,q,q^3;q^3)_n c^n d^n}.$$

If we let $c,d\to\infty$ in this identity we get that

$$a(q)=1+6\sum_{n=1}^{\infty}\frac{(-1)^n\,q^{(3n^2+n)/2}}{1-q^{3n}}-6\sum_{n=1}^{\infty}\frac{(-1)^n\,q^{(3n^2-n)/2}}{1-q^{3n}} -6\sum_{n=1}^{\infty}\frac{(1-q^{6n-1})q^{6n^2-2n}}{(1-q^{3n-1})(1-q^{3n})}+6\sum_{n=1}^{\infty}\frac{(1-q^{6n-2})q^{6n^2-4n}}{(1-q^{3n-2})(1-q^{3n})}. \tag{3.8}$$

A similar situation will hold for some of the other identities given below. Recall

$$\theta_3(q):=\sum_{n=-\infty}^{\infty}q^{n^2}=(-q,-q,q^2;q^2)_\infty=:\phi(q). \tag{3.9}$$

Corollary 6.

(3.10)

$$\theta_3(q)^2=1-4\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2-n}}{(q;q^4)_n(1-q^{4n})}+4\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+n}}{(q^3;q^4)_n(1-q^{4n})},$$
$$=1+4\sum_{n=1}^{\infty}\frac{(1-q^{8n-3})q^{4n^2-3n}}{(1-q^{4n-3})(1-q^{4n})}-4\sum_{n=1}^{\infty}\frac{(1-q^{8n-1})q^{4n^2-n}}{(1-q^{4n-1})(1-q^{4n})}.$$

For any values for c and d that do not make any denominator vanish,

$$\theta_3(q)^2=1+4\sum_{n=1}^{\infty}\frac{(q^4;q^4)_{n-1}(q^3/cd;q^4)_n\,(-1)^n\,q^{2n^2+n}}{(q^3/c,q^3/d,q^4;q^4)_n} -4\sum_{n=1}^{\infty}\frac{(q^4;q^4)_{n-1}(q/cd;q^4)_n\,(-1)^n\,q^{2n^2-n}}{(q/c,q/d,q^4;q^4)_n}$$
$$-4\sum_{n=1}^{\infty}\frac{(1-q^{8n-1})(q^4;q^4)_{n-1}(1/q,c,d;q^4)_n q^{4n^2+2n}}{(1-1/q)(q^3/c,q^3/d,q^3,q^4;q^4)_n c^n d^n}$$
$$+4\sum_{n=1}^{\infty}\frac{(1-q^{8n-3})(q^4;q^4)_{n-1}(1/q^3,c,d;q^4)_n q^{4n^2-2n}}{(1-1/q^3)(q/c,q/d,q,q^4;q^4)_n c^n d^n}. \tag{3.11}$$

Proof. By **Entry 8 (i)** in chapter 17 of [5],

$$\theta_3(q)^2=1+4\sum_{n=1}^{\infty}\frac{q^{4n-3}}{1-q^{4n-3}}-4\sum_{n=1}^{\infty}\frac{q^{4n-1}}{1-q^{4n-1}}.$$

We omit the remainder of the arguments, since they parallel those for the identities involving $a(q)$ above. □

If we let $c, d \to \infty$ in (3.11), we get the identity

$$\begin{aligned}(3.12)\quad \theta_3(q)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n\, q^{2n^2+n}}{1-q^{4n}} - 4\sum_{n=1}^{\infty} \frac{(-1)^n\, q^{2n^2-n}}{1-q^{4n}} \\ - 4\sum_{n=1}^{\infty} \frac{(1-q^{8n-1})q^{8n^2-2n}}{(1-q^{4n-1})(1-q^{4n})} + 4\sum_{n=1}^{\infty} \frac{(1-q^{8n-3})q^{8n^2-6n}}{(1-q^{4n-3})(1-q^{4n})}.\end{aligned}$$

Corollary 7. *If (α_n, β_n) is a Bailey pair with respect to a, then subject to suitable convergence conditions,*

$$(3.13)\qquad \sum_{n=1}^{\infty}(q^2;q^2)_{n-1}\,(-qa)^n\,\beta_n - \sum_{n=1}^{\infty} \frac{(q^2;q^2)_{n-1}\,(-qa)^n}{(q^2a^2;q^2)_n}\alpha_n = f_3(a,q),$$

where

$$\begin{aligned} f_3(a,q) &= -\sum_{n=1}^{\infty} \frac{(q\sqrt{a}, -q\sqrt{a}, a;q)_n(-q;q)_{n-1}q^{n(n+1)/2}a^n}{(\sqrt{a}, -\sqrt{a};q)_n(q^2a^2;q^2)_n(1-q^n)} \\ &= \sum_{n=1}^{\infty} \frac{(-q;q)_{n-1}\,(-qa)^n}{(qa;q)_n(1-q^n)} \\ &= -\sum_{n=1}^{\infty} \frac{aq^n}{1-a^2q^{2n}}. \end{aligned}$$

Proof. Let $z \to -1$ in (3.1) and simplify. □

One reason we single out this special case is that many theta products/series can also be expressed in terms of Lambert series of the type just above. We consider one example. Recall that

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

By **Entry 34 (p.284)** in chapter 36 of Ramanujan's notebooks (see [6, page 374]),

$$(3.14)\qquad q\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=1}^{\infty} \frac{q^{3n-2}}{1-q^{6n-4}} - \sum_{n=1}^{\infty} \frac{q^{3n-1}}{1-q^{6n-2}}.$$

Upon replacing q with q^3 and a with q^{-2} and then a with q^{-1} in Corollary 7 and combining the various serious appropriately, we get the following identities.

Corollary 8.

$$q\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=1}^{\infty}\frac{(1-q^{6n-2})(-q^3;q^3)_{n-1}q^{(3n^2-n)/2}}{(1-q^{3n-2})(1-q^{3n})(-q;q^3)_n} - \sum_{n=1}^{\infty}\frac{(1-q^{6n-1})(-q^3;q^3)_{n-1}q^{(3n^2+n)/2}}{(1-q^{3n-1})(1-q^{3n})(-q^2;q^3)_n}, \tag{3.15}$$

$$= \sum_{n=1}^{\infty}\frac{(-q^3;q^3)_{n-1}(-1)^n q^{2n}}{(1-q^{3n})(q^2;q^3)_n} - \sum_{n=1}^{\infty}\frac{(-q^3;q^3)_{n-1}(-1)^n q^{n}}{(1-q^{3n})(q;q^3)_n}. \tag{3.16}$$

If $(\alpha_n(a,q),\beta_n(a,q))$ *is a Bailey pair in which* a *is a free parameter, then*

$$\begin{aligned} q\frac{\psi^3(q^3)}{\psi(q)} &= \sum_{n=1}^{\infty}(q^6;q^6)_{n-1}(-q^2)^n\beta_n(1/q,q^3) - \sum_{n=1}^{\infty}(q^6;q^6)_{n-1}(-q)^n\beta_n(1/q^2,q^3) \\ &+ \sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(-q)^n}{(q^2;q^6)_n}\alpha_n(1/q^2,q^3) - \sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(-q^2)^n}{(q^4;q^6)_n}\alpha_n(1/q,q^3). \end{aligned} \tag{3.17}$$

If $d\neq q^{3n\pm1}$, *then*

$$\begin{aligned} q\frac{\psi^3(q^3)}{\psi(q)} &= \sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(q^2/d;q^6)_n(-q^2)^n}{(q^2;q^6)_n(q^2/d,q^3;q^3)_n} - \sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(q/d;q^6)_n(-q)^n}{(q;q^6)_n(q/d,q^3;q^3)_n} \\ &+ \sum_{n=1}^{\infty}\frac{(1-q^{12n-2})(q^6;q^6)_{2n-1}(1/q^2,d;q^6)_n q^{6n^2}}{(1-1/q^2)(q^2;q^6)_{2n}(q^4/d,q^6;q^6)_n d^n} \\ &- \sum_{n=1}^{\infty}\frac{(1-q^{12n-1})(q^6;q^6)_{2n-1}(1/q,d;q^6)_n q^{6n^2+3n}}{(1-1/q)(q^4;q^6)_{2n}(q^5/d,q^6;q^6)_n d^n}. \end{aligned} \tag{3.18}$$

$$\begin{aligned} q\frac{\psi^3(q^3)}{\psi(q)} &= \sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(-q^2)^n}{(q^2;q^6)_n(q^3;q^3)_n} - \sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(-q)^n}{(q;q^6)_n(q^3;q^3)_n} \\ &+ \sum_{n=1}^{\infty}\frac{(1-q^{12n-2})(q^6;q^6)_{2n-1}(1/q^2;q^6)_n(-1)^n q^{9n^2-3n}}{(1-1/q^2)(q^2;q^6)_{2n}(q^6;q^6)_n} \\ &- \sum_{n=1}^{\infty}\frac{(1-q^{12n-1})(q^6;q^6)_{2n-1}(1/q;q^6)_n(-1)^n q^{9n^2}}{(1-1/q)(q^4;q^6)_{2n}(q^6;q^6)_n}. \end{aligned} \tag{3.19}$$

Proof. Identities (3.15) - (3.17) follow directly from Corollary 8. The identity at (3.18) follows upon inserting the Bailey pair (see Corollary 2.13 in [11])

$$\alpha_{2r} = \frac{1-aq^{4r}}{1-a}\frac{\left(a,d;q^2\right)_r a^r q^{2r^2}}{\left(aq^2/d,q^2;q^2\right)_r d^r},$$
$$\alpha_{2r-1} = 0,$$
$$\beta_n = \frac{(aq/d;q^2)_n}{(aq;q^2)_n(aq/d,q;q)_n}, \quad \text{with respect to } a=a$$

into (3.17), and (3.19) is a consequence of letting $d\to\infty$ in (3.18). □

4. The Lambert series $\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}$ again

Define

$$L_a(q) := \sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}.$$

From Corollary 4 it can be seen that $L_a(q)$ can be variously represented as

$$\begin{aligned}(4.1)\quad L_a(q) &= \sum_{n=1}^{\infty}\frac{(1-aq^{2n})q^{n^2}a^n}{(1-aq^n)(1-q^n)}\\ &= -\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}\,(-a)^n}{(qa;q)_n(1-q^n)}\\ &= \sum_{n=1}^{\infty}\frac{(q;q)_{n-1}\,(-a)^n\,q^{n(n+1)/2}}{(qa;q)_n}\alpha_n - \sum_{n=1}^{\infty}(q;q)_{n-1}\,(-a)^n\,q^{n(n+1)/2}\beta_n,\end{aligned}$$

where (α_n,β_n) is a Bailey pair with respect to a. We subsequently noticed that it was possible to give two additional representations of $L_a(q)$.

Corollary 9.

$$\begin{aligned}L_a(q) &= \frac{-1}{(aq;q)_\infty}\sum_{n=1}^{\infty}\frac{n(-a)^nq^{n(n+1)/2}}{(q;q)_n}, \qquad (4.2)\\ &= (aq;q)_\infty\sum_{n=1}^{\infty}\frac{na^nq^n}{(q;q)_n}.\end{aligned}$$

Proof. Let $k\to 0$ and $z\to\infty$ in (1.3), and rearrange to get

$$\sum_{n=1}^{\infty}(yq;q)_{n-1}\left(\frac{-a}{y}\right)^n q^{n(n+1)/2}\beta_n - \frac{(qa/y;q)_\infty}{(qa;q)_\infty}\sum_{n=1}^{\infty}\frac{(yq;q)_{n-1}}{(qa/y;q)_n}\left(\frac{-a}{y}\right)^n q^{n(n+1)/2}\alpha_n = \frac{1}{1-y}\left(\frac{(qa/y;q)_\infty}{(qa;q)_\infty}-1\right),$$

where (α_n, β_n) is a Bailey pair with respect to a. The result now follows, upon comparison with Corollary 4, after using the q-binomial theorem (1.8) to expand in the infinite products inside the braces below as infinite series and then computing the limits:

$$\begin{aligned} L_a(q) &= -\lim_{y\to 1} \frac{1}{1-y} \frac{((qa/y;q)_\infty - (qa;q)_\infty)}{(qa;q)_\infty} \\ &= -\lim_{y\to 1} \frac{(qa/y;q)_\infty}{1-y}\left(\frac{1}{(qa;q)_\infty} - \frac{1}{(qa/y;q)_\infty}\right). \end{aligned}$$

□

These expressions for $L_a(q)$ may also be used to write any theta product that is expressible in terms of such Lambert series in terms of q-series similar to those in Corollary 9. Recall that $\theta(q)$ is defined at (3.9).

Corollary 10.

(4.3)

$$\begin{aligned} \frac{\theta^3(-q)}{\theta(-q)} &= 1 - \frac{6}{(-q;q^3)_\infty} \sum_{n=1}^{\infty} \frac{nq^{(3n^2+n)/2}}{(q^3;q^3)_n} + \frac{6}{(-q^2;q^3)_\infty} \sum_{n=1}^{\infty} \frac{nq^{(3n^2-n)/2}}{(q^3;q^3)_n}, \\ &= 1 + 6(-q;q^3)_\infty \sum_{n=1}^{\infty} \frac{n(-1)^n q^n}{(q^3;q^3)_n} - 6(-q^2;q^3)_\infty \sum_{n=1}^{\infty} \frac{n(-1)^n q^{2n}}{(q^3;q^3)_n}. \end{aligned}$$

Proof. By **Entry 18.2.16 (formula (1.21), p.353; formula (3.51), p.357)** in Ramanujan's Lost Notebook (see [3, page 405]),

$$\frac{\phi^3(-q)}{\phi(-q^3)} = 1 - 6\sum_{n=1}^{\infty} \frac{q^{3n-2}}{1+q^{3n-2}} + 6\sum_{n=1}^{\infty} \frac{q^{3n-1}}{1+q^{3n-1}}. \tag{4.4}$$

The proofs now follow as a consequence of Corollary 9. □

Remark: Ramanujan gives a number of other examples of theta products expressible as sums of Lambert series of the type considered in the present paper. The methods of the present paper could also be applied to those theta products, but we refrain from further examples, leaving these for the reader's own entertainment.

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