LIFTING BAILEY PAIRS TO WP-BAILEY PAIRS

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ABSTRACT. A pair of sequences $(\boldsymbol{\alpha}_n(a,k,q), \boldsymbol{\beta}_n(a,k,q))$ such that $\boldsymbol{\alpha}_0(a,k,q) = 1$ and

$$\boldsymbol{\beta}_n(a,k,q) = \sum_{j=0}^{n} \frac{(k/a;q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(aq;q)_{n+j}} \boldsymbol{\alpha}_j(a,k,q)$$

is termed a *WP-Bailey Pair*. Upon setting $k = 0$ in such a pair we obtain a *Bailey pair*.

In the present paper we consider the problem of "lifting" a Bailey pair to a WP-Bailey pair, and use some of the new WP-Bailey pairs found in this way to derive some new identities between basic hypergeometric series and new single sum- and double sum identities of the Rogers-Ramanujan-Slater type.

1. INTRODUCTION

A pair of sequences $\big(\alpha_n(a,q), \beta_n(a,q)\big)$ that satisfy $\alpha_0(a,q) = 1$ and

$$(1.1) \quad \beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}$$

$$= \frac{1}{(aq,q;q)_n} \sum_{r=0}^{n} \frac{(q^{-n};q)_r}{(aq^{n+1};q)_r}(-1)^r q^{nr-r(r-1)/2}\alpha_r(a,q)$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a_1, a_2, \ldots, a_j; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_j;q)_n,$$
$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots, \text{ and}$$
$$(a_1, a_2, \ldots, a_j; q)_\infty := (a_1;q)_\infty(a_2;q)_\infty \cdots (a_j;q)_\infty,$$

is termed a *Bailey pair relative to a*. Bailey [5, 6] showed that, for such a pair,

$$(1.2) \quad \sum_{n=0}^{\infty}(y,z;q)_n \left(\frac{aq}{yz}\right)^n \beta_n(a,q)$$

$$= \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(aq/y, aq/z; q)_n} \left(\frac{aq}{yz}\right)^n \alpha_n(a, q).$$

Slater, in [18] and [19], subsequently used this transformation of Bailey to derive 130 identities of the Rogers-Ramanujan type. Slater's method involved specializing y and z so that the series on right side of (1.2) became summable, using the Jacobi triple product identity.

$$(1.3) \qquad \sum_{n=-\infty}^{\infty} x^n q^{n^2} = (-q/x, -qx, q^2; q^2)_\infty.$$

In [2], Andrews extended the definition of a Bailey pair by setting $\boldsymbol{\alpha}_0(a, k, q) = 1$ and

(1.4)

$$\boldsymbol{\beta}_n(a, k, q) = \sum_{j=0}^{n} \frac{(k/a; q)_{n-j}(k; q)_{n+j}}{(q; q)_{n-j}(aq; q)_{n+j}} \boldsymbol{\alpha}_j(a, k, q)$$

$$= \frac{(k/a, k; q)_n}{(aq, q; q)_n} \sum_{j=0}^{n} \frac{(q^{-n}; q)_j(kq^n; q)_j}{(aq^{1-n}/k; q)_j(aq^{n+1}; q)_j} \left(\frac{qa}{k}\right)^j \boldsymbol{\alpha}_j(a, k, q).$$

Such a pair $(\boldsymbol{\alpha}_n(a, k, q), \boldsymbol{\beta}_n(a, k, q))$ is termed a *WP-Bailey pair*. Examples of WP Bailey pairs were previously given by Bressoud [7] and Singh [17]. Note that setting $k = 0$ in a WP-Bailey pair generates a standard Bailey pair, but it is not necessarily true that all standard Bailey pairs can be derived in this way (at least not if we insist that the $\boldsymbol{\beta}_n(a, k, q)$ in a WP-Bailey pair be in closed form).

We say that the Bailey pair $(\alpha_n(a, q), \beta_n(a, q))$ relative to a *lifts* to the WP-Bailey pair $(\boldsymbol{\alpha}_n(a, k, q), \boldsymbol{\beta}_n(a, k, q))$, or equivalently, that $(\boldsymbol{\alpha}_n(a, k, q), \boldsymbol{\beta}_n(a, k, q))$ is a *lift* of the pair $(\alpha_n(a, q), \beta_n(a, q))$, if

$$\boldsymbol{\alpha}_n(a, 0, q) = \alpha_n(a, q), \qquad \boldsymbol{\beta}_n(a, 0, q) = \beta_n(a, q), \qquad \forall\, n \geq 0.$$

Remark 1.1. Sometimes it will be convenient to suppress one or more of the parameters a, k, or q in the notation of ordinary and WP Bailey pairs. We shall, however, always distinguish between ordinary and WP Bailey pairs by denoting the latter in boldface.

The following (using slightly different notation) was proved in [14].

Theorem 1.2. *Let N be a positive integer. Suppose that $\boldsymbol{\alpha}_0 = 1$ and the sequences $\{\boldsymbol{\alpha}_n\}$ and $\{\boldsymbol{\beta}_n\}$ are related by*

$$\boldsymbol{\beta}_n = \sum_{j=0}^{n} \frac{(k/a; q)_{n-j}(k; q)_{n+j}}{(q; q)_{n-j}(aq; q)_{n+j}} \boldsymbol{\alpha}_j.$$

Then

$$(1.5) \quad \sum_{n=0}^{N} \frac{(1 - kq^{2n})(y, z, kaq^{N+1}/yz, q^{-N}; q)_n}{(1 - k)(kq/y, kq/z, yzq^{-N}/a, kq^{1+N}; q)_n} q^n \boldsymbol{\beta}_n$$

$$= \frac{(qk, qk/yz, qa/y, qa/z; q)_N}{(qk/y, qk/z, qa, qa/yz; q)_N}$$

$$\times \sum_{n=0}^{N} \frac{(y, z, kaq^{N+1}/yz, q^{-N}; q)_n}{(aq/y, aq/z, aq^{1+N}, yzq^{-N}/k; q)_n} \left(\frac{aq}{k}\right)^n \boldsymbol{\alpha}_n.$$

This turned out to be a re-formulation of one of the constructions Andrews [2] used to generate the *WP-Bailey lattice*, but we were unaware of the connection initially. Upon letting $N \to \infty$ we get the following result from [13].

Theorem 1.3. *Subject to suitable convergence conditions, if*

$$(1.6) \qquad \boldsymbol{\beta}_n = \sum_{r=0}^{n} \frac{(k/a; q)_{n-r}}{(q; q)_{n-r}} \frac{(k; q)_{n+r}}{(aq; q)_{n+r}} \boldsymbol{\alpha}_r,$$

then

$$(1.7) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z; q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z; q)_n} \left(\frac{qa}{yz}\right)^n \boldsymbol{\beta}_n =$$

$$\frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(qa/y, qa/z; q)_n} \left(\frac{qa}{yz}\right)^n \boldsymbol{\alpha}_n.$$

Notice that, if $\boldsymbol{\alpha}_n$ above is independent of k, then the series on the right sides of (1.2) and (1.7) are identical. Now suppose that a standard Bailey pair as in (1.1) lifts to a WP-Bailey pair in which $\boldsymbol{\alpha}_n$ is independent of k. If the standard Bailey pair gives rise to an identity of the Rogers-Ramanujan-Slater type, for certain choices of the parameters y and z, then it follows that the same choices for y and z will lead to a generalization of that identity, since the only occurrence of k on the right side of (1.7) is in the infinite product and the left side of (1.7) will thus also be an infinite product.

In [14] we found a WP-Bailey pair that is a lift of Slater's pair **F3**.

Theorem 1.4. *Define*

$$(1.8) \qquad \boldsymbol{\alpha}_n(1, k) = \begin{cases} 1, & n = 0, \\ q^{-n/2} + q^{n/2}, & n \geq 1, \end{cases}$$

$$\boldsymbol{\beta}_n(1, k) = \frac{(k\sqrt{q}, k; q)_n}{(\sqrt{q}, q; q)_n} q^{-n/2}.$$

Then $(\boldsymbol{\alpha}_n(1, k), \boldsymbol{\beta}_n(1, k))$ *satisfy* (1.4) *(with* $a = 1$*).*

The substitution of this pair into Theorem 1.2 leads to the following corollary.

Corollary 1.5.

$$(1.9) \quad {}_8\phi_7 \left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, y, z, k\sqrt{q}, kq^{1+N}/yz, q^{-N} \\ \sqrt{k}, -\sqrt{k}, qk/y, qk/z, \sqrt{q}, kq^{1+N}, yzq^{-N} \end{matrix} ; q, \sqrt{q} \right]$$

$$= \frac{(qk, qk/yz, q/y, q/z; q)_N}{(qk/y, qk/z, q, q/yz; q)_N}$$

$$\times \left(1 + \sum_{n=0}^{N} \frac{(1+q^n)(y, z, kq^{1+N}/yz, q^{-N}; q)_n}{(q/y, q/z, q^{1+N}, yzq^{-N}/k; q)_n} \left(\frac{\sqrt{q}}{k} \right)^n \right).$$

$$(1.10) \qquad \sum_{n=0}^{\infty} \frac{(1 - kq^{4n})(k; q)_{2n} q^{2n^2-n}}{(1-k)(q; q)_{2n}} = \frac{(kq^2; q^2)_\infty}{(q; q^2)_\infty}.$$

$$(1.11) \qquad \sum_{n=0}^{\infty} \frac{(1 - kq^{2n})(k; q)_n (-1)^n q^{n(n-1)/2}}{(1-k)(q; q)_n} = 0.$$

$$(1.12) \qquad \sum_{n=0}^{\infty} \frac{(1 - kq^{4n})(-q; q^2)_n (k; q)_{2n} q^{n^2-n}}{(1-k)(-kq; q^2)_n (q; q)_{2n}} = \frac{(kq^2, -1; q^2)_\infty}{(-kq, q; q^2)_\infty}.$$

Proof. The identity at (1.9) is immediate, while (1.10) follows upon letting $N \to \infty$, replacing q by q^2, letting $y, z \to \infty$ and finally using (1.3) to sum the right side. The identity at (1.11) is a consequence of letting $N \to \infty$, setting $y = \sqrt{q}$, letting $z \to \infty$ and again using (1.3) to sum the right side, and (1.12) follows similarly, except we set $y = -\sqrt{q}$ instead. □

Of course the last three identities are not new, as all can easily be seen to arise as special cases of (5.6). However, they do illustrate how a lift of a standard Bailey pair leads to generalizations of identities arising from this standard pair (the identities given by setting $k = 0$ in the corollary above).

The discovery of the WP-Bailey pair at (1.8) which is a lift of Slater's Bailey pair **F3**, together with the observation following Theorem 1.3, motivated us to investigate if other standard Bailey pairs could be lifted to a WP-Bailey pair, and to see what new transformations of basic hypergeometric series and what new identities of the Rogers-Ramanujan-Slater type would follow from these new WP-Bailey pairs.

It turned out that several of the lifts of Bailey pairs that we found could be derived as special cases of a result (see (2.13) below) of Singh [17] (see also Andrews and Berkovich [3]) . However, several others were not so easily explained, and in attempting to prove that some of these pairs that were found experimentally were indeed WP-Bailey pairs, we were led to consider various elementary ways of deriving new WP-Bailey pairs from existing pairs (ways that are different from those described by Andrews in [2]).

We also describe various ways in which double-sum identities of the Rogers -Ramanujan type identities may be easily derived from WP-Bailey pairs.

The second author defined three "multiparameter Bailey pairs" in [16], which, together with certain families of q-difference equations, "explain" more than half of the 130 Rogers-Ramanujan type identities in Slater's paper [19]. These multiparameter Bailey lift in a natural way to WP Bailey pairs, which in turn easily yield a variety of single and double-sum WP generalizations of Rogers-Ramanujan type identities.

2. WP-Bailey pairs arising from standard Bailey pairs

We first tried inserting the α_n that were part of Bailey pairs found by Slater [18, 19] into (1.4), and checking experimentally if the resulting $\beta_n(a, k)$ had closed forms. As a result, the following WP-Bailey pairs were found. The letter-number combination (e.g. **E7'**) refers to the standard pair in Slater's papers [18] and [19] recovered by setting $k = 0$. In all cases it is to be understood that $\boldsymbol{\alpha}_0 = \boldsymbol{\beta}_0 = 1$.

$$(2.1) \quad \boldsymbol{\alpha}_n(q, k) = \frac{(-1)^n(q^{-n} - q^{n+1})}{(1 - q)}, \qquad \textbf{(E7')}$$

$$\boldsymbol{\beta}_n(q, k) = \frac{(-1)^n(k^2; q^2)_n}{q^n(q^2; q^2)_n}.$$

$$(2.2) \quad \boldsymbol{\alpha}_n(1, k) = q^{-n/2} + q^{n/2}, \qquad \textbf{(F3')}$$

$$\boldsymbol{\beta}_n(1, k) = \frac{(k, kq^{1/2}; q)_n}{(q^{1/2}, q; q)_n} q^{-n/2}.$$

$$(2.3) \quad \boldsymbol{\alpha}_n(q, k) = \frac{q^{-n/2} + q^{n/2+1/2}}{1 + q^{1/2}}, \qquad \textbf{(F4')}$$

$$\boldsymbol{\beta}_n(q, k) = \frac{(k, kq^{-1/2}; q)_n}{(q^{3/2}, q; q)_n} q^{-n/2}.$$

$$(2.4) \quad \boldsymbol{\alpha}_n(1, k) = (-1)^n q^{-n^2/2} \left(q^{-3n/2} + q^{3n/2} \right), \qquad \textbf{(H3')}$$

$$\boldsymbol{\beta}_n(1, k) = \frac{(-1)^n(1 - kq^n + kq^{2n})(k; q)_n}{q^{(n^2+3n)/2}(q; q)_n}.$$

$$(2.5) \quad \boldsymbol{\alpha}_n(1, k) = (-1)^n q^{-n^2/2} \left(q^{-n/2} + q^{n/2} \right), \qquad \textbf{(H4')}$$

$$\boldsymbol{\beta}_n(1, k) = \frac{(-1)^n(k; q)_n}{q^{(n^2+n)/2}(q; q)_n}.$$

$$(2.6) \quad \boldsymbol{\alpha}_n(1, k) = q^{-n} - q^n, \qquad \textbf{(H5')}$$

$$\boldsymbol{\beta}_n(1, k) = \frac{(k, k; q)_n(1 - 2kq^n + kq^{2n})}{(1 - k)q^n(q, q; q)_n}.$$

(2.7) $\pmb{\alpha}_n(1,k) = 0,$ (**H6'**)

$$\pmb{\beta}_n(1,k) = \frac{(k,k;q)_n}{(q,q;q)_n}.$$

(2.8) $\pmb{\alpha}_n(1,k) = 2(-1)^n,$ (**H7'**)

$$\pmb{\beta}_n(1,k) = (-1)^n \frac{(k^2;q^2)_n}{(q^2;q^2)_n}.$$

(2.9) $\pmb{\alpha}_n(1,k) = (-1)^n(q^{-n} + q^n),$ (**H8'**)

$$\pmb{\beta}_n(1,k) = (-1)^n \frac{(k^2;q^2)_n(1 + kq^{2n})}{(1+k)q^n(q^2;q^2)_n}.$$

(2.10) $\pmb{\alpha}_n(q,k) = q^{-n} - q^{n+1},$ (**H12'**)

$$\pmb{\beta}_n(q,k) = \frac{(k;q)_n(k;q)_{n-1}(1 - kq^{n-1} - kq^n + kq^{2n})}{q^n(q,q^2;q)_n}.$$

(2.11) $\pmb{\alpha}_n(q,k) = 0,$ (**H13'**)

$$\pmb{\beta}_n(q,k) = \frac{(k,k/q;q)_n}{(q,q^2;q)_n}.$$

(2.12) $\pmb{\alpha}_n(1,q) = (-1)^n(1 + q^n)q^{(n^2-n)/2},$ (**H17'**)

$$\pmb{\beta}_n(1,q) = \frac{(-1)^n(k;q)_nq^{n(n-1)/2}k^n}{(q;q)_n}.$$

It turns out that eight of these are special cases of a more general WP-Bailey pair. In attempting to prove that these were indeed WP-Bailey pairs, we observed that the following WP-Bailey pair of Singh ([17] (see also Andrews and Berkovich [3]),

(2.13) $$\pmb{\alpha}'_n = \frac{(1 - aq^{2n})(a,c,d,a^2q/kcd;q)_n}{(1-a)(q,aq/c,aq/d,kcd/a;q)} \left(\frac{k}{a}\right)^n,$$

$$\pmb{\beta}'_n = \frac{(kc/a,kd/a,k,aq/cd;q)_n}{(aq/c,aq/d,q,kcd/a;q)_n},$$

is a lift of a standard Bailey pair of Slater [18, Equation (4.1), page 469],

(2.14) $$\alpha_n = \frac{(1 - aq^{2n})(a,c,d;q)_n}{(1-a)(aq/c,aq/d,q;q)_n} \left(\frac{-a}{cd}\right)^n q^{(n^2+n)/2},$$

$$\beta_n = \frac{(aq/cd;q)_n}{(aq/c,aq/d,q;q)_n}.$$

Remark 2.1. (1) We have replaced the ρ_1 and ρ_2 in [3] with c and d, to maintain consistency with Slater's notation in [18].

(2) Slater did not state the pair (2.14) explicitly as a Bailey pair, but instead listed many special cases (see Tables **B**, **F**, **E** and **H** in [18]).

By making the correct choices for a, c and d, it can be shown that the WP-Bailey pairs at (2.1), (2.2), (2.3), (2.5), (2.7), (2.8) , (2.11) and (2.12) are all special cases of (2.14). The proofs for the remaining pairs do not follow in the same way, because the corresponding standard Bailey pairs are *not* derived by substituting directly into (2.14). It is necessary to first derive some other preliminary results.

Theorem 2.2. *Suppose that* $(\boldsymbol{\alpha}_n(a), \boldsymbol{\beta}_n(a, k))$ *is a WP-Bailey pair such that* $\boldsymbol{\alpha}_n(a)$ *is independent of* k. *Then* $(\boldsymbol{\alpha}_n^*(a), \boldsymbol{\beta}_n^*(a, k))$ *is a WP-Bailey pair, where*

$$\boldsymbol{\alpha}_n^*(a) = (aq^n + q^{-n})\boldsymbol{\alpha}_n(a)$$

$$\boldsymbol{\beta}_n^*(a, k) = \frac{(1 + aq^{2n})\boldsymbol{\beta}_n(a, k) - (1 - k)\left(1 - \frac{k}{a}\right)\boldsymbol{\beta}_{n-1}(a, kq)}{q^n} - a\frac{(k, k/a; q)_n}{(aq, q; q)_n}.$$

Proof. From the definition of a WP-Bailey pair,

$$\boldsymbol{\beta}_n(a, kq) = \sum_{r=0}^{n} \frac{(kq; q)_{n+r}(kq/a; q)_{n-r}}{(aq; q)_{n+r}(q; q)_{n-r}}\boldsymbol{\alpha}_r(a)$$

$$= \sum_{r=0}^{n+1} \frac{(k; q)_{n+1+r}(k/a; q)_{n+1-r}(1 - aq^{n+1+r})(1 - q^{n+1-r})}{(1 - k)(1 - k/a)(aq; q)_{n+1+r}(q; q)_{n+1-r}}\boldsymbol{\alpha}_r(a)$$

$$= \frac{(1 + aq^{2n+2})}{(1 - k)(1 - k/a)} \sum_{r=0}^{n+1} \frac{(k; q)_{n+1+r}(k/a; q)_{n+1-r}}{(aq; q)_{n+1+r}(q; q)_{n+1-r}}\boldsymbol{\alpha}_r(a)$$

$$- \frac{q^{n+1}}{(1 - k)(1 - k/a)} \sum_{r=0}^{n+1} \frac{(k; q)_{n+1+r}(k/a; q)_{n+1-r}}{(aq; q)_{n+1+r}(q; q)_{n+1-r}}\boldsymbol{\alpha}_r(a)(aq^r + q^{-r})$$

$$= \frac{(1 + aq^{2n+2})}{(1 - k)(1 - k/a)}\boldsymbol{\beta}_{n+1}(a, k)$$

$$- \frac{q^{n+1}}{(1 - k)(1 - k/a)}\left[a\frac{(k; q)_{n+1}(k/a; q)_{n+1}}{(aq; q)_{n+1}(q; q)_{n+1}} + \boldsymbol{\beta}_{n+1}^*(a, k)\right].$$

The result follows upon replacing n with $n - 1$. $\square$

The following corollary is immediate, upon setting $k = 0$.

Corollary 2.3. *Suppose that (α_n, β_n) is a Bailey pair relative to a. Then so is (α_n^*, β_n^*), where*

$$\alpha_n^* = (aq^n + q^{-n})\alpha_n$$

$$\beta_n^* = \frac{(1 + aq^{2n})\beta_n - \beta_{n-1}}{q^n} - \frac{a}{(aq, q; q)_n}.$$

The following theorem follows directly from the definition of a WP-Bailey pair, so the proof is omitted.

Theorem 2.4. *Let $(\boldsymbol{\alpha}_n^{(1)}(a,k), \boldsymbol{\beta}_n^{(1)}(a,k))$ and $(\boldsymbol{\alpha}_n^{(2)}(a,k), \boldsymbol{\beta}_n^{(2)}(a,k))$ be WP-Bailey pairs and let c_1 and c_2 be constants. Then $(\boldsymbol{\alpha}_n(a,k), \boldsymbol{\beta}_n(a,k))$ is a WP-Bailey pair, where*

$$\boldsymbol{\alpha}_n(a,k) = c_1\boldsymbol{\alpha}_n^{(1)}(a,k) + c_2\boldsymbol{\alpha}_n^{(2)}(a,k),$$

$$\boldsymbol{\beta}_n(a,k) = c_1\boldsymbol{\beta}_n^{(1)}(a,k) + c_2\boldsymbol{\beta}_n^{(2)}(a,k) + (1 - c_1 - c_2)\frac{(k, k/a; q)_n}{(aq, q; q)_n}.$$

Note for later the special case $c_2 = 0$. The following corollary is immediate upon setting $k = 0$.

Corollary 2.5. *Let $(\alpha_n^{(1)}, \beta_n^{(1)})$ and $(\alpha_n^{(2)}, \beta_n^{(2)})$ be Bailey pairs relative to a, and let c_1 and c_2 be constants. Then (α_n, β_n) is a Bailey pair relative to a, where*

$$\alpha_n = c_1\alpha_n^{(1)} + c_2\alpha_n^{(2)},$$

$$\beta_n = c_1\beta_n^{(1)} + c_2\beta_n^{(2)} + \frac{(1 - c_1 - c_2)}{(aq, q; q)_n}.$$

Remark 2.6. Some the Bailey pairs derived by Slater [18, 19] follow from other pairs derived by her in [18, 19] as a result of Corollaries 2.3 and 2.5.

We are now in a position to prove that the pairs at (2.4), (2.6), (2.9) and (2.10) are indeed WP-Bailey pairs.

Corollary 2.7. *The pair of sequences $(\boldsymbol{\alpha}_n(1,k), \boldsymbol{\beta}_n(1,k))$, where*

$$\boldsymbol{\alpha}_n(1,k) = (-1)^n q^{-n^2/2}\left(q^{-3n/2} + q^{3n/2}\right), \tag{H3'}$$

$$\boldsymbol{\beta}_n(1,k) = \frac{(-1)^n(1 - kq^n + kq^{2n})(k; q)_n}{q^{(n^2+3n)/2}(q; q)_n},$$

is a WP-Bailey pair.

Proof. From what has been said previously, the pair at (2.5), namely

$$\boldsymbol{\alpha}_n^{(1)}(1,k) = (-1)^n q^{-n^2/2}\left(q^{-n/2} + q^{n/2}\right), \tag{H4'}$$

$$\boldsymbol{\beta}_n^{(1)}(1,k) = \frac{(-1)^n(k; q)_n}{q^{(n^2+n)/2}(q; q)_n}.$$

is a WP-Bailey pair. From Theorem 2.2, with $a = 1$,

$$\boldsymbol{\alpha}_n^{(2)}(1, k) = (q^n + q^{-n})\boldsymbol{\alpha}_n^{(1)}(1, k)$$

$$= (-1)^n q^{-n^2/2} \left(q^{-3n/2} + q^{3n/2} + q^{-n/2} + q^{n/2} \right),$$

$$\boldsymbol{\beta}_n^{(2)}(1, k) = \frac{(1 + q^{2n})\boldsymbol{\beta}_n^{(1)}(1, k) - (1 - k)(1 - k)\boldsymbol{\beta}_{n-1}^{(1)}(1, kq)}{q^n} - \frac{(k, k; q)_n}{(q, q; q)_n}$$

$$= \frac{(-1)^n (k; q)_n}{q^{(n^2+n)/2}(q; q)_n} \frac{1 + q^n - kq^n + kq^{2n}}{q^n} - \frac{(k, k; q)_n}{(q, q; q)_n},$$

is a WP-Bailey pair. The result now follows from Theorem 2.4, upon setting $a = 1$, $c_1 = -1$, $c_2 = 1$ and letting $(\boldsymbol{\alpha}_n^{(1)}(1, k), \boldsymbol{\beta}_n^{(1)}(1, k))$ and $(\boldsymbol{\alpha}_n^{(2)}(1, k), \boldsymbol{\beta}_n^{(2)}(1, k))$ be as stated above. $\square$

We next prove that the pair at (2.9) is a WP-Bailey pair.

Corollary 2.8. *The pair of sequences* $(\boldsymbol{\alpha}_n(1, k), \boldsymbol{\beta}_n(1, k))$, *where*

$$\boldsymbol{\alpha}_n(1, k) = (-1)^n (q^{-n} + q^n), \tag{$\boldsymbol{H8'}$}$$

$$\boldsymbol{\beta}_n(1, k) = (-1)^n \frac{(k^2; q^2)_n (1 + kq^{2n})}{(1 + k)q^n (q^2; q^2)_n}.$$

is a WP-Bailey pair.

Proof. The proof is similar to that for (2.4) in the corollary above, except we start with

$$\boldsymbol{\alpha}_n^{(1)}(1, k) = 2(-1)^n, \tag{$\boldsymbol{H7'}$}$$

$$\boldsymbol{\beta}_n^{(1)}(1, k) = (-1)^n \frac{(k^2; q^2)_n}{(q^2; q^2)_n}.$$

Theorem 2.2 gives

$$\boldsymbol{\alpha}_n^{(2)}(1, k) = 2(-1)^n (q^n + q^{-n}),$$

$$\boldsymbol{\beta}_n^{(2)}(1, k) = (-1)^n \frac{(k^2; q^2)_n (2 + 2q^{2n})}{q^n (1 + k)(q^2; q^2)_n} - \frac{(k, k; q)_n}{(q, q; q)_n},$$

is a WP-Bailey pair. The result follows once again from Theorem 2.4, upon setting $c_2 = 1/2$ and $c_1 = 0$. $\square$

We next give proofs for the two remaining two pairs at (2.6) and (2.10).

Corollary 2.9. *The pairs of sequences*

$$\boldsymbol{\alpha}_n(1, k) = q^{-n} - q^n, \tag{$\boldsymbol{H5'}$}$$

$$\boldsymbol{\beta}_n(1, k) = \frac{(k, k; q)_n (1 - 2kq^n + kq^{2n})}{(1 - k)q^n (q, q; q)_n}.$$

and

$$\boldsymbol{\alpha}_n(q, k) = q^{-n} - q^{n+1}, \qquad\qquad (\boldsymbol{H12'})$$

$$\boldsymbol{\beta}_n(q, k) = \frac{(k; q)_n (k; q)_{n-1}(1 - kq^{n-1} - kq^n + kq^{2n})}{q^n (q, q^2; q)_n}.$$

are WP-Bailey pairs.

Proof. Let $c \to \infty$ (or $c \to 0$) and set $d = q$ in (2.13) to get that

$$\alpha_n(a, k) = \frac{q^{-n} - aq^n}{1 - a},$$

$$\beta_n(a, k) = \frac{(kq/a, k; q)_n}{(a, q; q)_n q^n}$$

is a WP-Bailey pair. Now apply Theorem 2.4 with $c_1 = 1 - a$ and $c_2 = 0$ (with $(\boldsymbol{\alpha}_n^{(1)}(a, k), \boldsymbol{\beta}_n^{(1)}(a, k)) = (\boldsymbol{\alpha}_n^{(2)}(a, k), \boldsymbol{\beta}_n^{(2)}(a, k)) = (\boldsymbol{\alpha}_n(a, k), \boldsymbol{\beta}_n(a, k)))$ to get that

$$\boldsymbol{\alpha}_n^*(a, k) = q^{-n} - aq^n,$$

$$\boldsymbol{\beta}_n^*(a, k) = \frac{(kq/a, k; q)_n}{(aq; q)_{n-1}(q; q)_n q^n} + a\frac{(k, k/a; q)_n}{(aq, q; q)_n},$$

is a WP-Bailey pair. The pairs in the statement of the corollary are, respectively, the cases $a = 1$ and $a = q$. $\qquad\square$

Remark 2.10. If lifts exist for the remaining Bailey pairs found by Slater [18, 19], finding them will likely prove more difficult, as experiment seems to indicate that the α_n are *not* independent of k.

For completeness sake, we include the following theorem in this section, as it gives yet another way of deriving new Bailey pairs from existing Bailey pairs. We first note the identities

$$(2.15) \qquad 1 = \frac{1 - aq^{n+r+1}}{1 - aq^{2r+1}} - aq^{2r+1}\frac{1 - q^{n-r}}{1 - aq^{2r+1}}.$$

$$(2.16) \qquad 1 = q^{-n+r}\frac{1 - aq^{n+r+1}}{1 - aq^{2r+1}} - q^{-n+r}\frac{1 - q^{n-r}}{1 - aq^{2r+1}}.$$

Theorem 2.11. *Suppose* $(\alpha_n(a, q), \beta_n(a, q))$ *is a Bailey pair with respect to* a. *Then so are the pairs* $(\alpha_n^*(a, q), \beta_n^*(a, q))$ *and* $(\alpha_n^\dagger(a, q), \beta_n^\dagger(a, q))$, *where* $\alpha_0^*(a, q) = \beta_0^*(a, q) = \alpha_0^\dagger(a, q) = \beta_0^\dagger(a, q) = 1$, *and for* $n > 0$,

$$\alpha_n^*(a, q) = (1 - aq)\left(\frac{\alpha_n(aq, q)}{1 - aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}(aq, q)}{1 - aq^{2n-1}}\right),$$

$$\beta_n^*(a, q)) = \beta_n(aq, q),$$

$$\alpha_n^\dagger(a, q) = (1 - aq)\left(q^n\frac{\alpha_n(aq, q)}{1 - aq^{2n+1}} - q^{n-1}\frac{\alpha_{n-1}(aq, q)}{1 - aq^{2n-1}}\right),$$

$$\beta_n^\dagger(a, q)) = q^n \beta_n(aq, q).$$

Proof. From the definition of a Bailey pair,

$$\beta_n(aq, q) = \sum_{r=0}^{n} \frac{\alpha_r(aq, q)}{(q; q)_{n-r}(aq^2; q)_{n+r}}$$

$$= \sum_{r=0}^{n-1} \frac{\alpha_r(aq, q)}{(q; q)_{n-r}(aq^2; q)_{n+r}} + \frac{\alpha_n(aq, q)}{(aq^2; q)_{2n}}$$

$$= (1 - aq) \sum_{r=0}^{n-1} \frac{\alpha_r(aq, q)}{(q; q)_{n-r}(aq; q)_{n+r+1}} + \frac{\alpha_n(aq, q)}{(aq^2; q)_{2n}}$$

$$= (1 - aq) \sum_{r=0}^{n-1} \frac{\alpha_r(aq, q)}{(1 - aq^{2r+1})(q; q)_{n-r}(aq; q)_{n+r}}$$

$$- (1 - aq) \sum_{r=0}^{n-1} \frac{aq^{2r+1}\alpha_r(aq, q)}{(1 - aq^{2r+1})(q; q)_{n-r-1}(aq; q)_{n+r+1}} + \frac{\alpha_n(aq, q)}{(aq^2; q)_{2n}}$$

$$= (1 - aq) \sum_{r=0}^{n-1} \frac{\alpha_r(aq, q)}{(1 - aq^{2r+1})(q; q)_{n-r}(aq; q)_{n+r}}$$

$$- (1 - aq) \sum_{r=1}^{n} \frac{aq^{2r-1}\alpha_{r-1}(aq, q)}{(1 - aq^{2r-1})(q; q)_{n-r}(aq; q)_{n+r}} + \frac{\alpha_n(aq, q)}{(aq^2; q)_{2n}}.$$

The next-to-last equality follows from (2.15) and the result now follows for $(\alpha_n^*(a, q), \beta_n^*(a, q))$. The result for $(\alpha_n^\dagger(a, q), \beta_n^\dagger(a, q))$ follows similarly, except we use (2.16) at the next to last step. $\qquad\square$

Corollary 2.12. *The pairs* $(\alpha_n^*(a, q), \beta_n^*(a, q))$ *and* $(\alpha_n^\dagger(a, q), \beta_n^\dagger(a, q))$ *are Bailey pairs with respect to a, where* $\alpha_0^*(a, q) = \beta_0^*(a, q) = \alpha_0^\dagger(a, q) = \beta_0^\dagger(a, q) = 1$, *and for* $n > 0$,

$$\alpha_n^*(a, q) = \frac{(aq, c, d; q)_n}{(aq^2/c, aq^2/d, q; q)_n} \left(\frac{-aq}{cd}\right)^n q^{(n^2+n)/2}$$

$$- aq^{2n-1} \frac{(aq, c, d; q)_{n-1}}{(aq^2/c, aq^2/d, q; q)_{n-1}} \left(\frac{-aq}{cd}\right)^{n-1} q^{(n^2-n)/2},$$

$$\beta_n^*(a, q)) = \frac{(aq^2/cd; q)_n}{(aq^2/c, aq^2/d, q; q)_n},$$

$$\alpha_n^\dagger(a, q) = q^n \frac{(aq, c, d; q)_n}{(aq^2/c, aq^2/d, q; q)_n} \left(\frac{-aq}{cd}\right)^n q^{(n^2+n)/2}$$

$$- q^{n-1} \frac{(aq, c, d; q)_{n-1}}{(aq^2/c, aq^2/d, q; q)_{n-1}} \left(\frac{-aq}{cd}\right)^{n-1} q^{(n^2-n)/2},$$

$$\beta_n^\dagger(a, q)) = q^n \frac{(aq^2/cd; q)_n}{(aq^2/c, aq^2/d, q; q)_n}.$$

Proof. This follows directly from applying Theorem 2.11 to the Bailey pair at (2.14). $\qquad\square$

3. Dual WP-Bailey pairs

We next consider a natural pairing of WP-Bailey pairs. We first recall that

$$(3.1) \qquad (a; 1/q)_n = \frac{(-a)^n (1/a; q)_n}{q^{n(n-1)/2}}.$$

Andrews showed in [1] that if $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair relative to a, then $(\alpha_n^*(a, q), \beta_n^*(a, q))$ is also a Bailey pair relative to a, where

$$\alpha_n^*(a, q) = a^n q^{n^2} \alpha_n(1/a, 1/q),$$

$$\beta_n^*(a, q) = a^{-n} q^{-n^2-n} \beta_n(1/a, 1/q).$$

The pair $(\alpha_n^*(a, q), \beta_n^*(a, q))$ is called *the dual* of $(\alpha_n(a, q), \beta_n(a, q))$. Note that the dual of $(\alpha_n^*(a, q), \beta_n^*(a, q))$ is $(\alpha_n(a, q), \beta_n(a, q))$. As an example, the dual of the Bailey pair

$$\alpha_n(1, q) = (-1)^n q^{3n^2/2} \left(q^{-3n/2} + q^{3n/2} \right), \qquad\qquad \textbf{(B2)}$$

$$\beta_n(1, q) = \frac{q^n}{(q; q)_n}.$$

is the Bailey pair

$$\alpha_n(1, q) = (-1)^n q^{-n^2/2} \left(q^{-3n/2} + q^{3n/2} \right), \qquad\qquad \textbf{(H3)}$$

$$\beta_n(1, q) = \frac{(-1)^n}{q^{(n^2+3n)/2}(q; q)_n}.$$

This concept of duality can be extended to WP-Bailey pairs.

Theorem 3.1. *Suppose* $(\boldsymbol{\alpha}_n(a, k, q), \boldsymbol{\beta}_n(a, k, q))$ *is a WP-Bailey pair. Then* $(\boldsymbol{\alpha}_n^*(a, k, q), \boldsymbol{\beta}_n^*(a, k, q))$ *is also a WP-Bailey pair, where*

$$\boldsymbol{\alpha}_n^*(a, k, q) = \boldsymbol{\alpha}_n(1/a, 1/k, 1/q),$$

$$\boldsymbol{\beta}_n^*(a, k, q) = \left(\frac{k}{aq} \right)^{2n} \boldsymbol{\beta}_n(1/a, 1/k, 1/q).$$

Proof. Replace a by $1/a$, k by $1/k$ and q by $1/q$ in (1.4) and use (3.1) to simplify the resulting expression. $\qquad\square$

As with standard Bailey pairs, we refer to the pair $(\boldsymbol{\alpha}_n^*(a, k, q), \boldsymbol{\beta}_n^*(a, k, q))$ in Theorem 3.1 as *the dual* of $(\boldsymbol{\alpha}_n(a, k, q), \boldsymbol{\beta}_n(a, k, q))$. Note that, as above, the dual of $(\boldsymbol{\alpha}_n^*(a, k, q), \boldsymbol{\beta}_n^*(a, k, q))$ is $(\boldsymbol{\alpha}_n(a, k, q), \boldsymbol{\beta}_n(a, k, q))$.

We also remark that it is possible to use these duality constructions to derive new Bailey- or WP-Bailey pairs.

Corollary 3.2. *The pair of sequences $(\boldsymbol{\alpha}_n^*(1,k), \boldsymbol{\beta}_n^*(1,k))$ is a WP-Bailey pair, where*

$$(3.2) \qquad \boldsymbol{\alpha}_n^*(1,k) = (-1)^n q^{n^2/2} \left(q^{-3n/2} + q^{3n/2} \right),$$

$$\boldsymbol{\beta}_n^*(1,k) = \frac{(-1)^n (1 - q^n + kq^{2n})(k;q)_n k^{n-1} q^{(n^2-3n)/2}}{(q;q)_n}.$$

Proof. The pair at (3.2) is the dual of the pair at (2.4):

$$\boldsymbol{\alpha}_n(1,k) = (-1)^n q^{-n^2/2} \left(q^{-3n/2} + q^{3n/2} \right), \qquad \qquad \textbf{(H3')}$$

$$\boldsymbol{\beta}_n(1,k) = \frac{(-1)^n (1 - kq^n + kq^{2n})(k;q)_n}{q^{(n^2+3n)/2}(q;q)_n}.$$

$\square$

Remark 3.3. The Bailey pair derived from (3.2) by setting $k = 0$ does not appear in Slater's lists of Bailey pairs in [18, 19].

4. Basic Hypergeometric series identities and identities of the Rogers-Ramanujan-Slater type

Each of the WP-Bailey pairs found above may be substituted into (1.5) and (1.7), leading to possibly new identities between basic hypergeometric series and/or new identities of the Rogers-Ramanujan type. We illustrate this by considering the WP-Bailey pair from (3.2).

Corollary 4.1. *Let N be a positive integer and suppose $k \neq 0$.*

$$(4.1)$$

$$\sum_{n=0}^{N} \frac{\left(q\sqrt{k}, -q\sqrt{k}, k, y, z, \frac{kq^{N+1}}{yz}, \frac{1-\sqrt{1-4k}}{2}q, \frac{1+\sqrt{1-4k}}{2}q, q^{-N}; q \right)_n}{\left(\sqrt{k}, -\sqrt{k}, \frac{kq}{y}, \frac{kq}{z}, yzq^{-N}, kq^{1+N}, \frac{1-\sqrt{1-4k}}{2}, \frac{1+\sqrt{1-4k}}{2}, q; q \right)_n} (-k)^n q^{\frac{n^2-n}{2}}$$

$$= \frac{\left(qk, \frac{qk}{yz}, \frac{q}{y}, \frac{q}{z}; q \right)_N}{\left(\frac{qk}{y}, \frac{qk}{z}, q, \frac{q}{yz}; q \right)_N}$$

$$\times \left(1 + \sum_{n=1}^{N} \frac{\left(y, z, \frac{kq^{N+1}}{yz}, q^{-N}; q \right)_n}{\left(\frac{q}{y}, \frac{q}{z}, \frac{yzq^{-N}}{k}, q^{1+N}; q \right)_n} \left(\frac{-1}{k} \right)^n q^{(n^2-n)/2} \left(1 + q^{3n} \right) \right).$$

$$(4.2) \quad \sum_{n=0}^{\infty} \frac{\left(q\sqrt{k}, -q\sqrt{k}, k, \frac{1-\sqrt{1-4k}}{2}q, \frac{1+\sqrt{1-4k}}{2}q; q \right)_n (q;q^2)_n k^n q^{(n^2-3n)/2}}{\left(\sqrt{k}, -\sqrt{k}, \frac{1-\sqrt{1-4k}}{2}, \frac{1+\sqrt{1-4k}}{2}, q; q \right)_n (k^2q; q^2)_n}$$

$$= \frac{1+k}{q} \frac{\left(q^2k^2, ; q^2 \right)_\infty}{(qk^2; q^2)_\infty}.$$

$$(4.3) \quad \sum_{n=0}^{\infty} \frac{\left(q\sqrt{k}, -q\sqrt{k}, k, \frac{1-\sqrt{1-4k}}{2}q, \frac{1+\sqrt{1-4k}}{2}q; q\right)_n (-k)^n q^{3(n^2-n)/2}}{\left(\sqrt{k}, -\sqrt{k}, \frac{1-\sqrt{1-4k}}{2}, \frac{1+\sqrt{1-4k}}{2}, q; q\right)_n} = 0.$$

$$(4.4) \quad \sum_{n=0}^{\infty} \frac{\left(q\sqrt{k}, -q\sqrt{k}, k, \sqrt{q}, \frac{1-\sqrt{1-4k}}{2}q, \frac{1+\sqrt{1-4k}}{2}q; q\right)_n k^n q^{n^2-3n/2}}{\left(\sqrt{k}, -\sqrt{k}, k\sqrt{q}, \frac{1-\sqrt{1-4k}}{2}, \frac{1+\sqrt{1-4k}}{2}, q; q\right)_n}$$
$$= q^{-1/2} \frac{(kq; q)_\infty}{(k\sqrt{q}; q)_\infty}.$$

Proof. The identity at (4.1) follows upon substituting the pair from (3.2) into (1.5), setting $a = 1$, and using the fact that

$$1 - q^n + kq^{2n} = k \frac{\left(\frac{1-\sqrt{1-4k}}{2}q, \frac{1+\sqrt{1-4k}}{2}q; q\right)_n}{\left(\frac{1-\sqrt{1-4k}}{2}, \frac{1+\sqrt{1-4k}}{2}; q\right)_n}.$$

Next, let $N \to \infty$ in (4.1) to get that

$$(4.5) \quad \sum_{n=0}^{\infty} \frac{\left(q\sqrt{k}, -q\sqrt{k}, k, y, z, \frac{1-\sqrt{1-4k}}{2}q, \frac{1+\sqrt{1-4k}}{2}q; q\right)_n}{\left(\sqrt{k}, -\sqrt{k}, \frac{kq}{y}, \frac{kq}{z}, \frac{1-\sqrt{1-4k}}{2}, \frac{1+\sqrt{1-4k}}{2}, q; q\right)_n} \left(\frac{-k}{yz}\right)^n q^{\frac{n^2-n}{2}}$$
$$= \frac{\left(qk, \frac{qk}{yz}, \frac{q}{y}, \frac{q}{z}; q\right)_\infty}{\left(\frac{qk}{y}, \frac{qk}{z}, q, \frac{q}{yz}; q\right)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(y, z; q)_n}{\left(\frac{q}{y}, \frac{q}{z}; q\right)_n} \left(\frac{-1}{yz}\right)^n q^{(n^2-n)/2} \left(1 + q^{3n}\right)\right).$$

The identities at (4.2), (4.3) and (4.4) follow, respectively, from letting $(y, z) \to (\sqrt{q}, -\sqrt{q})$, $(y, z) \to (\infty, \infty)$ and $(y, z) \to (\sqrt{q}, \infty)$, and using the Jacobi triple product identity (1.3) to sum the resulting series on the right sides. $\qquad\square$

5. Double-sum identities of the Rogers-Ramanujan-Slater type

If $(\alpha_n)_{n \geq 0}$ is any sequence with $\alpha_0 = 1$, then trivially

$$\left(\alpha_n, \sum_{j=0}^{n} \frac{(k/a; q)_{n-j}(k; q)_{n+j}}{(q; q)_{n-j}(aq; q)_{n+j}} \alpha_j\right)$$

is a WP-Bailey pair. If this pair is substituted into (1.7), then after switching the order of summation on the left side and re-indexing, we get the following theorem.

Theorem 5.1. *Let $(\alpha_n)_{n \geq 0}$ be any sequence with $\alpha_0 = 1$. Then, subject to suitable convergence conditions,*

$$(5.1) \quad \sum_{j,n=0}^{\infty} \frac{(1 - kq^{2n+2j})(k/a; q)_n(k; q)_{n+2j}(y, z; q)_{n+j}}{(1 - k)(q; q)_n(aq; q)_{n+2j}(qk/y, qk/z; q)_{n+j}} \left(\frac{qa}{yz}\right)^{n+j} \alpha_j$$

$$= \frac{(qk, qk/yz, qa/y, qa/z; q)_\infty}{(qk/y, qk/z, qa, qa/yz; q)_\infty} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(qa/y, qa/z; q)_n} \left(\frac{qa}{yz}\right)^n \alpha_n.$$

This last equation can be used to derive, almost trivially, a large number of double-sum series = product identities. Firstly, any identity on the Slater list can be extended to a double-sum identity involving the free parameter k, and which reverts back to the original single-sum identity upon setting $k = 0$. This is simply done by inserting the sequence α_n from the same Bailey pair (α_n, β_n), and making the same choices for y and z, as Slater did to derive the original identity. Secondly, we can choose (α_n) so that the series on the right of (5.1) becomes one of the series in an identity on the Slater list, so that the right side once again can be expressed as an infinite product. Thirdly, we can choose α_n so that the series on the right side becomes summable via the Jacobi triple product identity or the quintuple product identity. We illustrate each of these methods of generating a double-sum series = product by giving some examples.

We first consider the standard pair **B1** from Slater's paper [18]. This pair has

$$(5.2) \qquad \alpha_j = \begin{cases} 1, & j = 0, \\ q^{3j^2/2 - j/2}(-1)^j(1 + q^j), & j > 0, \end{cases}$$

and leads to the first Rogers-Ramanujan identity.

$$(5.3) \qquad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_\infty}.$$

Corollary 5.2. *For $k \neq 1$,*

$$(5.4) \quad \sum_{j,n=0}^{\infty} \frac{(1 - kq^{2n+2j})(1 + q^j)(k; q)_n(k; q)_{n+2j} q^{(5j^2-j)/2 + 2nj + n^2}(-1)^j}{(1 - k)(q; q)_n(q; q)_{n+2j}}$$

$$= \frac{(kq; q)_\infty}{(q, q^4; q^5)_\infty} + \frac{(kq; q)_\infty}{(q; q)_\infty}.$$

Proof. First, set $a = 1$ and let $y, z \to \infty$ in (5.1) to get

$$(5.5) \quad \sum_{j,n=0}^{\infty} \frac{(1 - kq^{2n+2j})(k; q)_n(k; q)_{n+2j}}{(1 - k)(q; q)_n(q; q)_{n+2j}} q^{(n+j)^2} \alpha_j = \frac{(qk; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} q^{n^2} \alpha_n.$$

Next, substitute for (α_n), writing $\alpha_0 = 1$ as

$$\alpha_0 = 1 = q^{3(0^2)/2 - 0/2}(-1)^0(1 + q^0) - 1,$$

and use (1.3) to sum the right side, to get

$$\sum_{j,n=0}^{\infty} \frac{(1-kq^{2n+2j})(1+q^j)(k;q)_n(k;q)_{n+2j}q^{(5j^2-j)/2+2nj+n^2}(-1)^j}{(1-k)(q;q)_n(q;q)_{n+2j}}$$

$$-\sum_{n=0}^{\infty} \frac{(1-kq^{2n})(k,k;q)_n q^{n^2}}{(1-k)(q,q;q)_n} = \frac{(qk;q)_\infty}{(q;q)_\infty}(q^2,q^3,q^5;q^5)_\infty = \frac{(kq;q)_\infty}{(q,q^4;q^5)_\infty}.$$

The fact that

$$\sum_{n=0}^{\infty} \frac{(1-kq^{2n})(k,k;q)_n q^{n^2}}{(1-k)(q,q;q)_n} = \frac{(kq;q)_\infty}{(q;q)_\infty}$$

follows as a special case of the following identity, which is a special case of an identity due to Jackson [11] (set $a = k$, $b = k$ and let $c, d \to \infty$).

(5.6)

$$_6\phi_5 \left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{array} ; q, \frac{aq}{bcd} \right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}.$$

$\square$

Corollary 5.3.

(5.7) $$\sum_{j,n=0}^{\infty} \frac{(1-kq^{2n+2j})(k/a;q)_n(k;q)_{n+2j}(yq^j, zq^j;q)_n(qa/y, qa/z;q)_j}{(1-k)(q;q)_n(aq;q)_{n+2j}(qk/y, qk/z;q)_{n+j}(q;q)_j}$$

$$\times \left(\frac{qa}{yz}\right)^n q^{j^2} = \frac{(qk, qk/yz, qa/y, qa/z;q)_\infty}{(qk/y, qk/z, qa, qa/yz;q)_\infty} \frac{1}{(q,q^4;q^5)_\infty}.$$

Proof. Set

$$\alpha_n = \frac{(qa/y, qa/z;q)_n}{(y,z;q)_n}\left(\frac{yz}{qa}\right)^n \frac{q^{n^2}}{(q;q)_n}$$

in (5.1), so that the series on the left side of (5.1) becomes the series on the right side of (5.3). $\square$

   Corollary 5.2 is an extension of the first Rogers-Ramanujan identity, since setting $k = 0$ recovers this identity, after some series manipulations. It is possible to generalize the identity in Corollary 5.2 as follows (Corollary 5.2 is the case $s = 5/2$, $r = 1/2$ of the following corollary).

Corollary 5.4. *Let s be a positive rational number and r a rational number. For $k \neq 1$,*

(5.8) $$\sum_{j,n=0}^{\infty} \frac{(1-kq^{2n+2j})(1+q^{2rj})(k;q)_n(k;q)_{n+2j}q^{sj^2-jr+2nj+n^2}(-1)^j}{(1-k)(q;q)_n(q;q)_{n+2j}}$$

$$= \frac{(kq;q)_\infty(q^{s-r}, q^{s+r}, q^{2s}; q^{2s})_\infty}{(q;q)_\infty} + \frac{(kq;q)_\infty}{(q;q)_\infty}.$$

Proof. The proof is similar to the proof of Corollary 5.2, except we make the substitution

$$\alpha_n = q^{(s-1)n^2 - nr}(-1)^n(1 + q^{2nr})$$

in (5.5). □

6. WP Generalizations of the Multiparameter Bailey Pairs

In [16], the second author showed that more than half of the identities in Slater's list could be recovered by specializing parameters in just three general Bailey pairs together with some q-difference equations.

The *standard multiparameter Bailey pair (SMBP)* [16] is defined as follows:

Let

(6.1)

$$\alpha_n^{(d,e,h)}(a,q) := \frac{(-1)^{n/d}a^{(h/d-1)n/e}q^{(h/d-1+1/2d)n^2/e - n/2e}(1 - a^{1/e}q^{2n/e})}{(1 - a^{1/e})(q^{d/e};q^{d/e})_{n/d}}$$
$$\times (a^{1/e};q^{d/e})_{n/d}\,\chi(d\mid n),$$

where

$$\chi(P(n,d)) = \begin{cases} 1 & \text{if } P(n,d) \text{ is true,} \\ 0 & \text{if } P(n,d) \text{ is false,} \end{cases}$$

and let $\beta_n^{(d,e,h)}(a,q)$ be determined by (1.1).

The *Euler multiparameter Bailey pair (EMBP)* is given by

(6.2) $$\widetilde{\alpha}_n^{(d,e,h)}(a,q) := q^{n(d-n)/2de}a^{-n/de}\frac{(-a^{1/e};q^{d/e})_{n/d}}{(-q^{d/e};q^{d/e})_{n/d}}\alpha_n^{(d,e,h)}(a,q)$$

with $\widetilde{\beta}_n^{(d,e,h)}(a,q)$ determined by (1.1), and the *Jackson-Slater multiparameter Bailey pair (JSMBP)* by

(6.3) $$\bar{\alpha}_n^{(d,e,k)}(a,q) := (-1)^{n/d}q^{-n^2/2de}\frac{(q^{d/2e};q^{d/e})_{n/d}}{(a^{1/e}q^{d/2e};q^{d/e})_{n/d}}\alpha_n^{(d,e,h)}(a,q)$$

with $\bar{\beta}_n^{(d,e,h)}(a,q)$ determined by (1.1).

Clearly each of the α's in (6.1)–(6.3) could be inserted into (1.4) instead of (1.1) to produce WP generalizations of the multiparameter Bailey pairs.

Let us therefore define

(6.4) $$\boldsymbol{\alpha}_n^{(d,e,h)}(a,k,q) := \alpha_n^{(d,e,h)}(a,q),$$

(6.5) $$\widetilde{\boldsymbol{\alpha}}_n^{(d,e,h)}(a,k,q) := \widetilde{\alpha}_n^{(d,e,h)}(a,q),$$

(6.6) $$\bar{\boldsymbol{\alpha}}_n^{(d,e,k)}(a,k,q) := \bar{\alpha}_n^{(d,e,k)}(a,q),$$

and employ (1.4) to obtain the following corresponding $\boldsymbol{\beta}$'s:

(6.7) $$\boldsymbol{\beta}_n^{(d,e,h)}(a^e,k,q^e) = \frac{(k,k/a^e;q^e)_n}{(q^e,a^eq^e;q^e)_n}$$

$$\times \sum_{r=0}^{\lfloor n/d \rfloor} \frac{(a;q^d)_r(1-aq^{2dr})(q^{-en},kq^{en};q^e)_{dr}}{(q^d;q^d)_r(1-a)(a^eq^{e(n+1)},k^{-1}a^eq^{e(1-n)};q^e)_{dr}}$$

$$\times (-1)^r a^{(h-d+ed)r} k^{-dr} q^{(2h-2d+1)dr^2/2+(2e-1)dr/2}$$

(6.8) $\quad \widetilde{\boldsymbol{\beta}}_n^{(d,e,h)}(a^e,k,q^e) = \dfrac{(k,k/a^e;q^e)_n}{(q^e,a^eq^e;q^e)_n}$

$$\times \sum_{r=0}^{\lfloor n/d \rfloor} \frac{(a^2;q^{2d})_r(1-aq^{2dr})(q^{-en},kq^{en};q^e)_{dr}}{(q^{2d};q^{2d})_r(1-a)(a^eq^{e(n+1)},k^{-1}a^eq^{e(1-n)};q^e)_{dr}}$$

$$\times (-1)^r a^{(h-d-1+ed)r} k^{-dr} q^{(h-d)dr^2+edr}$$

(6.9) $\quad \bar{\boldsymbol{\beta}}_n^{(d,e,h)}(a^e,k,q^e) = \dfrac{(k,k/a^e;q^e)_n}{(q^e,a^eq^e;q^e)_n}$

$$\times \sum_{r=0}^{\lfloor n/d \rfloor} \frac{(a,q^{d/2};q^d)_r(1-aq^{2dr})(q^{-en},kq^{en};q^e)_{dr}}{(q^d,aq^{d/2};q^d)_r(1-a)(a^eq^{e(n+1)},k^{-1}a^eq^{e(1-n)};q^e)_{dr}}$$

$$\times a^{(h-d+ed)r} k^{-dr} q^{(h-d)dr^2+(2e-1)dr/2}.$$

Thus each of $\left(\boldsymbol{\alpha}_n^{(d,e,h)}(a,k;q), \boldsymbol{\beta}_n^{(d,e,h)}(a,k;q) \right)$, $\left(\widetilde{\boldsymbol{\alpha}}_n^{(d,e,h)}(a,k,q), \right.$ $\left. \widetilde{\boldsymbol{\beta}}_n^{(d,e,h)}(a,k,q) \right)$, and $\left(\bar{\boldsymbol{\alpha}}_n^{(d,e,h)}(a,k,q), \bar{\boldsymbol{\beta}}_n^{(d,e,h)}(a,k,q) \right)$ is a WP Bailey pair. Note that the series in each of (6.7)–(6.9) may be expressed as a limiting case of a very-well-poised $_{t+1}\phi_t$ basic hypergeometric series, where

$$t = |2h - 2d + 1| + 2ed + 2,$$

and as such is either summable or transformable via standard formulas found in Gasper and Rahman's book [8].

Proposition 6.1. *The WP multiparameter Bailey pairs* $\left(\boldsymbol{\alpha}_n^{(1,1,1)}(a,k,q), \boldsymbol{\beta}_n^{(1,1,1)}(a,k,q) \right)$, $\left(\widetilde{\boldsymbol{\alpha}}_n^{(1,1,1)}(a,k,q), \widetilde{\boldsymbol{\beta}}_n^{(1,1,1)}(a,k,q) \right)$, *and* $\left(\bar{\boldsymbol{\alpha}}_n^{(1,1,1)}(a,k,q), \bar{\boldsymbol{\beta}}_n^{(1,1,1)}(a,k,q) \right)$ *are given by*

(6.10) $\quad \boldsymbol{\alpha}_n^{(1,1,1)}(a,k,q) = \dfrac{(-1)^n q^{n(n-1)/2}(1-aq^{2n})(a;q)_n}{(1-a)(q;q)_n}$

(6.11) $\quad \boldsymbol{\beta}_n^{(1,1,1)}(a,k,q) = \dfrac{(-1)^n k^n q^{\binom{n}{2}}(k;q)_n}{a^n(q;q)_n}$

(6.12) $\quad \widetilde{\boldsymbol{\alpha}}_n^{(1,1,1)}(a,k,q) = \dfrac{(-1)^n a^{-n}(1-aq^{2n})(a^2;q^2)_n}{(1-a)(q^2;q^2)_n}$

(6.13) $\quad \widetilde{\boldsymbol{\beta}}_n^{(1,1,1)}(a,k,q) = \dfrac{(-1)^n a^{-n}(k^2;q^2)_n}{(q^2;q^2)_n}$

(6.14) $\bar{\boldsymbol{\alpha}}_n^{(1,1,1)}(a,k,q) = \dfrac{q^{-n/2}(1-aq^{2n})(a,\sqrt{q};q)_n}{(1-a)(q,a\sqrt{q})_n}$

(6.15) $\bar{\boldsymbol{\beta}}_n^{(1,1,1)}(a,k,q) = \dfrac{(k,k\sqrt{q}/a;q)_n}{(q,a\sqrt{q};q)_n}q^{-n/2}$

Proof. Each of the α's is a direct substitution into the definition with $d = e = h = 1$. Each of the β's follows from Jackson's summation of a very-well-poised $_6\phi_5$ [8, Eq. (II.21)].   □

We may now use these WP Bailey pairs to derive WP generalizations of Rogers-Ramanujan-Slater type identities.

Corollary 6.2.

(6.16) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{2n})(k;q)_n}{(1-k)(q;q)_n}(-1)^n k^n q^{n(3n-1)/2} = (kq;q)_\infty$

(6.17) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{4n})(-q,k;q^2)_n}{(1-k)(-kq,q^2;q^2)_n}(-1)^n k^n q^{2n^2-n} = \frac{(kq^2;q^2)_\infty}{(-kq;q^2)_\infty}$

(6.18) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{2n})(-1,k;q)_n}{(1-k)(-kq,q;q)_n}(-1)^n k^n q^{n^2} = \frac{(kq;q)_\infty}{(-kq;q)_\infty}$

(6.19) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{2n})(k^2;q^2)_n}{(1-k)(q^2;q^2)_n}(-1)^n q^{n^2} = (kq;q)_\infty(q;q^2)_\infty$

(6.20) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{4n})(-q;q^2)_n(k^2;q^4)_n}{(1-k)(-kq;q^2)_n(q^4;q^4)_n}(-1)^n q^{n^2} = \frac{(kq^2;q^2)_\infty(q;q)_\infty}{(-kq;q^2)_\infty(q^4;q^4)_\infty}$

(6.21) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{4n})(k;q)_{2n}}{(1-k)(q;q)_{2n}}q^{2n^2-n} = \frac{(kq^2;q^2)_\infty}{(q;q^2)_\infty}$

(6.22) $\displaystyle\sum_{n=0}^{\infty}\frac{(1-kq^{4n})(-q;q^2)_n(k;q)_{2n}}{(1-k)(-kq;q^2)_n(q;q)_{2n}}q^{n^2-n} = \frac{(kq^2,-1;q^2)_\infty}{(-kq,q;q^2)_\infty}$

Proof. To obtain (6.16), insert (6.10)–(6.11) into (1.7) with $a = 1$ and $y, z \to \infty$. To obtain (6.17), insert (6.10)–(6.11) into (1.7) with $a = 1$, $y = -\sqrt{q}$ and $z \to \infty$. To obtain (6.18), insert (6.10)–(6.11) into (1.7) with $a = 1$, $y = -q$ and $z \to \infty$. To obtain (6.19), insert (6.12)–(6.13) into (1.7) with $a = 1$ and $y, z \to \infty$. To obtain (6.20), insert (6.12)–(6.13) into (1.7) with $a = 1$, $y = -\sqrt{q}$ and $z \to \infty$. To obtain (6.21), insert (6.14)–(6.15) into (1.7) with $a = 1$ and $y, z \to \infty$. To obtain (6.22), insert (6.14)–(6.15) into (1.7) with $a = 1$, $y = -\sqrt{q}$ and $z \to \infty$.   □

Remark 6.3. Setting $k = 0$ in (6.19) recovers Eq. (3) of Slater [19]. We had obtained (6.21) and (6.22) earlier via another method (see Eqs. (1.10) and (1.12)). Setting $k = 0$ in (6.21) recovers Eq. (9) of Slater [19], an identity originally due to Jackson [12, p. 179, 3 lines from bottom].

If any of d, e, or h is greater than 1, then the representation of the β as a finite product times a very-well-poised ${}_{t+1}\phi_t$ will have $t > 6$, will thus not be summable. Accordingly, the WP-Rogers-Ramanujan-Slater type identities obtained from these will involve double sums.

Let us now consider the case which leads to a generalization of the first Rogers-Ramanujan identity.

Proposition 6.4. $\left(\boldsymbol{\alpha}_n^{(1,1,2)}(a,k,q), \boldsymbol{\beta}_n^{(1,1,2)}(a,k,q)\right)$ *is given by*

$$(6.23) \qquad \boldsymbol{\alpha}_n^{(1,1,2)}(a,k,q) = \frac{(-1)^n a^n q^{n(3n-1)/2}(1 - aq^{2n})(a;q)_n}{(1-a)(q;q)_n}$$

$$(6.24) \qquad \boldsymbol{\beta}_n^{(1,1,2)}(a,k,q) = (k;q)_n \sum_{j=0}^{n} \frac{(-1)^j k^j q^{\binom{j}{2}+nj}(k/a;q)_{n-j}}{(q;q)_j (q;q)_{n-j}}$$

Proof. The $\alpha_n^{(1,1,2)}(a,k,q)$ follows by direct substitution into (6.4). The $\beta_n^{(1,1,2)}(a,k,q)$ follows by specializing (6.7) and applying Watson's q-analog of Whipple's theorem [8, Eq. (III.18)]. $\qquad\square$

Corollary 6.5 (a WP-generalization of the first Rogers-Ramanujan identity)**.**
(6.25)
$$\sum_{n,j\geq 0} \frac{(1 - kq^{2n+2j})q^{n^2+3nj+j(5j-1)/2}(-1)^j k^j (k;q)_n (k;q)_{n+j}}{(1-k)(q;q)_j (q;q)_n} = \frac{(kq;q)_\infty}{(q, q^4; q^5)_\infty}.$$

Proof. Insert $\left(\boldsymbol{\alpha}_n^{(1,1,2)}(a,k,q), \boldsymbol{\beta}_n^{(1,1,2)}(a,k,q)\right)$ into (1.7) with $a = 1$ and $y, z \to \infty$, interchange the order of summation on the left hand side and apply Jacobi's triple product identity (1.3) on the right hand side. $\qquad\square$

Corollary 6.6 (a WP-generalization of the first Göllnitz-Gordon identity)**.**

(6.26)
$$\sum_{n,j\geq 0} \frac{(-1)^j k^j (1 - kq^{4n+4j})q^{n^2+4nj+4j^2-j}(-q;q^2)_{n+j}(k;q^2)_n (k;q^2)_{n+j}}{(1-k)(-kq;q^2)_{n+j}(q^2;q^2)_j (q^2;q^2)_n}$$
$$= \frac{(kq^2;q^2)_\infty}{(-kq;q^2)_\infty (q, q^4, q^7; q^8)_\infty}.$$

Proof. Insert $\left(\boldsymbol{\alpha}_n^{(1,1,2)}(a,k,q), \boldsymbol{\beta}_n^{(1,1,2)}(a,k,q)\right)$ into (1.7) with $a = 1$, $y = -\sqrt{q}$ and $z \to \infty$, interchange the order of summation on the left hand side and apply Jacobi's triple product identity (1.3) on the right hand side. Finally, replace q by q^2 throughout. $\qquad\square$

Remark 6.7. By sending $k \to 0$ in (6.26), we recover Identity (36) of Slater [19]. An equivalent analytic identity was recorded by Ramanujan in the lost notebook [4]. This identity became well-known after being interpreted partition theoretically by Göllnitz [9] and Gordon [10].

Many additional WP-analogs of known Rogers-Ramanujan type identities could easily be derived using the WP-multiparameter Bailey pairs. We shall content ourselves here with several examples where the series expressions are not too complicated. Each of the following identities can be proved by inserting an appropriate WP-Bailey pair into a limiting case of (1.7), and applying Jacobi's triple product identity (1.3).

A WP-generalization of the first Rogers-Selberg mod 7 identity [15, p. 339]; cf. [19, Eq. (33)]:

$$(6.27) \quad \sum_{n,r\geq 0} \frac{(1-kq^{4n+4r})(k;q^2)_{n+2r}}{(1-k)(q^2;q^2)_r(-q;q)_{2r}(q^2;q^2)_n}(-1)^n k^n q^{n(3n-1)+4nr+2r^2}$$
$$= \frac{(kq^2;q^2)_\infty(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.$$

A WP-generalization of the Jackson-Slater identity [12, p. 170, 5th Eq.]; cf. [19, Eq. (39)]:

$$(6.28) \quad \sum_{n,r\geq 0} \frac{(1-kq^{4n+4r})(k;q^2)_{n+2r}(kq;q^2)_{n+r}(q;q^2)_r}{(1-k)(q^2;q^2)_r(q;q^2)_{n+r}(kq;q^2)_r(q^2;q^2)_n}(-1)^r q^{2n^2-n+2nr+r^2}$$
$$= \frac{(kq^2;q^2)_\infty(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}.$$

A WP-generalization of Bailey's mod 9 identity [5, p. 422, Eq. (1.8)], cf. [19, Eq. (42)]:

$$(6.29) \quad \sum_{n,r\geq 0} \frac{(1-kq^{6n+6r})(k;q^3)_{n+2r}(q;q)_{3r}}{(1-k)(q^3;q^3)_n(q^3;q^3)_{2r}(q^3;q^3)_r}(-1)^n k^n q^{3n(3n-1)/2+6nr+3r^2}$$
$$= \frac{(kq^3;q^3)_\infty(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}.$$

A WP-generalization of Rogers's mod 14 identity [15, p. 341, Ex. 2]; cf. [19, Eq. (61)]:

$$(6.30) \quad \sum_{n,r\geq 0} \frac{(1-kq^{2n+2r})(k;q)_{n+2r}}{(1-k)(q;q^2)_r(q;q)_r(q;q)_n}(-1)^n k^n q^{n(3n-1)/2+2nr+r^2}$$
$$= \frac{(kq;q)_\infty(q^6,q^8,q^{14};q^{14})_\infty}{(q;q)_\infty}.$$

7. SLATER REVISITED

It would interesting to lift all the Bailey pairs found by Slater to WP-Bailey pairs, but at present we do not have a general method that will allow us to do this.

As was noted earlier, it is likely that finding lifts of the other Bailey pairs will be more difficult, as experimentation suggests that the sequence α_n will be dependent on the parameter k.

It is hoped that some of the results in the present paper might interest others in the search for lifts of the remaining Bailey pairs in the Slater papers [18, 19].

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