POWERS OF A MATRIX AND COMBINATORIAL IDENTITIES

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ABSTRACT. In this article we obtain a general polynomial identity in k variables, where $k \geq 2$ is an arbitrary positive integer.

We use this identity to give a closed-form expression for the entries of the powers of a $k \times k$ matrix.

Finally, we use these results to derive various combinatorial identities.

1. Introduction

In [4], the second author had observed that the following 'curious' polynomial identity holds:

$$\sum (-1)^i \binom{n-i}{i} (x+y)^{n-2i} (xy)^i = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n.$$

The proof was simply observing that both sides satisfied the same recursion. He had also observed (but not published the result) that this recursion defines in a closed form the entries of the powers of a 2×2 matrix in terms of its trace and determinant and the entries of the original matrix. The first author had independently discovered this fact and derived several combinatorial identities as consequences [2].

In this article, for a general k, we obtain a polynomial identity and show how it gives a closed-form expression for the entries of the powers of a $k \times k$ matrix. From these, we derive some combinatorial identities as consequences.

2. Main Results

Throughout the paper, let K be any fixed field of characteristic zero. We also fix a positive integer k. The main results are the following two theorems:

Theorem 1. *Let $x_1, \cdots, x_k$ be independent variables and let $s_1, \cdots, s_k$ denote the various symmetric polynomials in the x_i's of degrees $1, 2 \cdots, k$ respectively. Then, in the polynomial ring $K[x_1, \cdots, x_k]$, for each positive*

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integer n, one has the identity

$$\sum_{r_1+\cdots+r_k=n} x_1^{r_1}x_2^{r_2}\cdots x_k^{r_k} = \sum_{2i_2+3i_3+\cdots+ki_k\leq n} c(i_2,\cdots,i_k,n)s_1^{n-2i_2-3i_3-\cdots-ki_k}(-s_2)^{i_2}s_3^{i_3}\cdots((-1)^{k-1}s_k)^{i_k},$$

where

$$c(i_2,\cdots,i_k,n) = \frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-(ki_k)!}.$$

Theorem 2. *Suppose $A \in M_k(K)$ and let*

$$T^k - s_1T^{k-1} + s_2T^{k-2} + \cdots + (-1)^k s_k\, I$$

denote its characteristic polynomial. Then, for all $n \geq k$, one has

$$A^n = b_{k-1}A^{k-1} + b_{k-2}A^{k-2} + \cdots + b_0\, I,$$

where

$$\begin{aligned} b_{k-1} &= a(n-k+1),\\ b_{k-2} &= a(n-k+2) - s_1a(n-k+1),\\ &\vdots\\ b_1 &= a(n-1) - s_1a(n-2) + \cdots + (-1)^{k-2}s_{k-2}a(n-k+1),\\ b_0 &= a(n) - s_1a(n-1) + \cdots + (-1)^{k-1}s_{k-1}a(n-k+1)\\ &= (-1)^{k-1}s_ka(n-k). \end{aligned}$$

and

$$a(n) = c(i_2,\cdots,i_k,n)s_1^{n-i_2-2i_3-\cdots-(k-1)i_k}(-s_2)^{i_2}s_3^{i_3}\cdots((-1)^{k-1}s_k)^{i_k},$$

with

$$c(i_2,\cdots,i_k,n) = \frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-(ki_k)!}.$$

as in Theorem 1.

Proof of Theorems 1 and 2. In Theorem 1, if $a(n)$ denotes either side, it is straightforward to verify that

$$a(n) = s_1a(n-1) - s_2a(n-2) + \cdots + (-1)^{k-1}s_ka(n-k).$$

Theorem 2 is a consequence of Theorem 1 on using induction on n.

□

The special cases $k = 2$ and $k = 3$ are worth noting for it is easier to derive various combinatorial identities from them.

Corollary 1. *(i) Let $A \in M_3(K)$ and let $X^3 = tX^2 - sX + d$ denote the characteristic polynomial of A. Then, for all $n \geq 3$,*

$$A^n = a_{n-1}A + a_{n-2}Adj(A) + (a_n - ta_{n-1})\, I, \tag{2.1}$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} t^{n-2i-3j} s^i d^j$$

for $n > 0$ and $a_0 = 1$.

(ii) Let $B \in M_2(K)$ and let $X^2 = t\,X - d$ denote the characteristic polynomial of B. Then, for all $n \geq 2$,

$$B^n = b_n I + b_{n-1} Adj(B)$$

for all $n \geq 2$, where

$$b_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i.$$

Corollary 2. *Let $\theta \in K$, $B \in M_2(K)$ and t denote the trace and d the determinant of B. We have the following identity in $M_2(K)$:*

$$(a_{n-1} - \theta a_{n-2})B + (a_n - (\theta + t)a_{n-1} + \theta a_{n-2} t)I$$
$$= y_{n-1}B + (y_n - t\,y_{n-1})I,$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + t)^{n-2i-3j} (\theta t + d)^i (\theta d)^j$$

and

$$y_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i.$$

In particular, for any $\theta \in K$, one has

$$b_n - (\theta + 1)b_{n-1} + \theta b_{n-2} = 1,$$

where

$$b_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + 2)^{n-2i-3j} (1 + 2\theta)^i \theta^j.$$

Corollary 3. *The numbers $c_n = \sum_{2i+3j=n} (-1)^i \binom{i+j}{j} 2^i 3^j$ satisfy*

$$c_n + c_{n-1} - 2c_{n-2} = 1.$$

Proof. This is the special case of Corollary 2 where we take $\theta = -2$. Note that the sum defining c_n is over only those i, j for which $2i + 3j = n$. □

Note than when $k = 3$, Theorem 1 can be rewritten as follows:

Theorem 3. *Let n be a positive integer and x, y, z be indeterminates. Then*

(2.2)
$$\sum_{2i+3j\leq n}(-1)^i\binom{i+j}{j}\binom{n-i-2j}{i+j}(x+y+z)^{n-2i-3j}(xy+yz+zx)^i(xyz)^j$$
$$=\frac{x\,y\,\left(x^{n+1}-y^{n+1}\right)-x\,z\,\left(x^{n+1}-z^{n+1}\right)+y\,z\,\left(y^{n+1}-z^{n+1}\right)}{(x-y)\,(x-z)\,(y-z)}.$$

Proof. In Corollary 1, let
$$A=\begin{pmatrix} x+y+z & 1 & 0\\ -x\,y-x\,z-y\,z & 0 & 1\\ x\,y\,z & 0 & 0\end{pmatrix}.$$
Then $t=x+y+z$, $s=xy+xz+yz$ and $d=xyz$. It is easy to show (by first diagonalizing A) that the $(1,2)$ entry of A^n equals the right side of (2.2), with $n+1$ replaced by n, and the $(1,2)$ entry on the right side of (2.1) is a_{n-1}. □

Corollary 4. *Let x and z be indeterminates and n a positive integer. Then*
$$\sum_{2i+3j\leq n}(-1)^i\binom{i+j}{j}\binom{n-i-2j}{i+j}(2x+z)^{n-2i-3j}(x^2+2xz)^i(x^2z)^j$$
$$=\frac{x^{2+n}+n\,x^{1+n}\,(x-z)-2\,x^{1+n}\,z+z^{2+n}}{(x-z)^2}.$$

Proof. Let $y\to x$ in Theorem 3. □

Some interesting identities can be derived by specialising the variables in Theorem 1. For instance, in [5], it was noted that Binet's formula for the Fibonacci numbers is a consequence of Theorem 1 for $k=2$. Here is a generalization.

Corollary 5. *(Generalization of Binet's formula)*
Let the numbers $F_k(n)$ be defined by the recursion
$$F_k(0)=1, F_k(r)=0\forall r<0,$$
$$F_k(n)=F_k(n-1)+F_k(n-2)+\cdots+F_k(n-k).$$
Then, we have
$$F_k(n)=\sum_{2i_2+\cdots+ki_k\leq n}\frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_1!i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-ki_k)!}.$$
Further, this equals $\sum_{r_1+\cdots+r_k=n}\lambda_1^{r_1}\cdots\lambda_k^{r_k}$ where $\lambda_i, 1\leq i\leq k$ are the roots of the equation $T^k-T^{k-1}-T^{k-2}-\cdots-1=0$.

Proof. The recursion defining $F_k(n)$'s corresponds to the case $s_1=-s_2=\cdots=(-1)^{k-1}s_k=1$ of the theorem. □

Corollary 6.

$$\sum c(i_2,\cdots,i_k,n)k^n\prod_{j=2}^{k}\left((-1)^{j-1}k^{-j}\binom{k}{j}\right)^{i_j}=\binom{n+k-1}{k}.$$

where

$$c(i_2,\cdots,i_k,n)=\frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-ki_k)!}.$$

Proof. Take $x_i=1$ for all i in Theorem 1. The left side of Theorem 1 is simply the sum $\sum_{r_1+\cdots+r_k=n}1$. □

From Theorem3 we have the following binomial identities as special cases.

Proposition 1. *(i) Let λ be the unique positive real number satisfying $\lambda^3=\lambda+1$. Let x,y denote the complex conjugates such that $xy=\lambda, x+y=\lambda^2$, and let $z=-\frac{1}{\lambda}$. Then,*

$$\sum_{2i+3j\le n}(-1)^j\binom{n-2j}{j}=\sum_{r+s+t=n}x^ry^sz^t$$
$$=\frac{x\,y\,\left(x^{n+1}-y^{n+1}\right)-x\,z\,\left(x^{n+1}-z^{n+1}\right)+y\,z\,\left(y^{n+1}-z^{n+1}\right)}{(x-y)\,(x-z)\,(y-z)}.$$

(ii)

$$\sum_{2i+3j\le n}(-1)^j\binom{i+j}{j}\binom{n-i-2j}{i+j}=[(n+2)/2].$$

(iii)

$$\sum\binom{n-2j}{j}(-4)^j3^{n-3j}=\frac{(3n+4)2^{n+1}+(-1)^n}{9}.$$

(iv)

$$\sum\binom{n-2j}{j}3^{n-3j}(-2)^j$$
$$=\frac{(1+\sqrt{3})^{n+1}-(1-\sqrt{3})^{n+1}}{2\sqrt{3}}+\frac{(1+\sqrt{3})^{n+1}+(1-\sqrt{3})^{n+1}}{6}-\frac{1}{3}.$$

3. Commutating Matrices

In this section we derive various combinatorial identities by writing a general 3×3 Matrix A as a product of commuting matrices.

Proposition 2. *Let A be an arbitrary 3×3 matrix with characteristic equation $x^3-tx^2+s\,x-d=0$, $d\neq0$. Suppose p is arbitrary, with*

$p^3+p^2t+ps+d\neq 0$, $p\neq 0,\,-t$. *If* n *is a positive integer, then*

$$A^n=\left(\frac{p\,d}{p^3+p^2t+sp+d}\right)^n\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k} \times\left(\frac{-p(p+t)^2}{d}\right)^j\left(\frac{-(p+t)}{p}\right)^k\left(\frac{-A}{p+t}\right)^r. \tag{3.1}$$

Proof. This follows from the identity

$$A=\frac{-1}{p^3+p^2t+sp+d}\left(pA^2-Ap(p+t)-d\,I\right)\left(A+p\,I\right),$$

after raising both sides to the n-th power and collecting powers of A. Note that the two matrices $pA^2-Ap(p+t)-d\,I$ and $A+p\,I$ commute. □

Corollary 7. *Let* p, x, y *and* z *be indeterminates and let* n *be a positive integer. Then*

$$\begin{aligned}\sum_{r=0}^{3\,n}\sum_{j=0}^{n}\sum_{k=0}^{n}&\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}\left(\frac{p(p+x+y+z)^2}{xyz}\right)^j\\&\times\left(\frac{p+x+y+z}{p}\right)^k\frac{x\,y\,\left(x^r-y^r\right)-x\,z\,\left(x^r-z^r\right)+y\,z\,\left(y^r-z^r\right)}{(p+x+y+z)^r}\\&=\left(x\,y\,\left(x^n-y^n\right)-x\,z\,\left(x^n-z^n\right)+y\,z\,\left(y^n-z^n\right)\right)\\&\qquad\times\left(\frac{p^3+p^2\,\left(x+y+z\right)+p\,\left(x\,y+x\,z+y\,z\right)+x\,y\,z}{p\,x\,y\,z}\right)^n.\end{aligned}$$

Proof. Let A be the matrix from Theorem 3 and compare $(1,1)$ entries on both sides of (3.1). □

Corollary 8. *Let* p, x *and* z *be indeterminates and let* n *be a positive integer. Then*

$$\begin{aligned}\sum_{r=0}^{3\,n}\sum_{j=0}^{n}\sum_{k=0}^{n}&\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}\left(\frac{p(p+2x+z)^2}{x^2z}\right)^j\\&\times\left(\frac{p+2x+z}{p}\right)^k\frac{r\,x^{1+r}-x^r\,z-r\,x^r\,z+z^{1+r}}{(p+2x+z)^r}\\&=\left(n\,x^{1+n}-x^n\,z-n\,x^n\,z+z^{1+n}\right)\\&\qquad\times\left(\frac{p^3+p^2\,\left(2x+z\right)+p\,\left(x^2+2x\,z\right)+x^2\,z}{p\,x^2\,z}\right)^n.\end{aligned}$$

Proof. Divide both sides in the corollary above by $x-y$ and let $y\to x$. □

Corollary 9. *Let p and x be indeterminates and let n be a positive integer. Then*

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}\left(\frac{p(p+3x)^2}{x^3}\right)^j \times\left(\frac{p+3x}{p}\right)^k\frac{r\ (1+r)\ x^{-1+r}}{2(p+3x)^r} =\frac{n\ (1+n)\ x^{-1+n}}{2}\left(\frac{(p+x)^3}{p\,x^3}\right)^n.$$

Proof. Divide both sides in the corollary above by $(x-z)^2$ and let $z\to x$. □

Corollary 10. *Let p be an indeterminate and let n be a positive integer. Then*

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+3)^{2j+k-r}\frac{r\ (1+r)}{2} =\frac{n\ (1+n)}{2}\frac{(p+1)^{3n}}{p^n}.$$

Proof. Replace p by $p\,x$ in the corollary above and simplify. □

Various combinatorial identities can be derived from Theorem 3 by considering matrices A such that particular entries in A^n have a simple closed form. We give four examples.

Corollary 11. *Let n be a positive integer.*
(i) If $p\neq 0,-1$, then

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+3)^{2j+k-r}r=n\frac{(1+p)^{3n}}{p^n}.$$

(ii) Let F_n denote the n-th Fibonacci number. If $p\neq 0,-1,\phi$ or $1/\phi$ (where ϕ is the golden ratio, then

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{k+r}p^{j-k}(p+2)^{2j+k-r}F_r =F_n\frac{(1+p)^n\left(-1+p+p^2\right)^n}{(-p)^n}.$$

(iii) If $p \neq 0, -1$ *or* -2*, then*

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+4)^{2j+k-r}2^{-j}(2^r-1) = (2^n-1)\left(\frac{(1+p)^2(p+2)}{2p}\right)^n.$$

(iv) If $p \neq 0, -1, -g$ *or* $-h$ *and* $gh \neq 0$*, then*

$$\sum_{r=0}^{3n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}\binom{j}{r-j-k}(-1)^{j+k+r}p^{j-k}(p+1+g+h)^{2j+k-r} \times \frac{g^r+h^r}{(g\,h)^j} = (g^n+h^n)\left(\frac{(1+p)(g+p)(h+p)}{ghp}\right)^n.$$

Proof. The results follow from considering the $(1,2)$ entries on both sides in Theorem 3 for the matrices

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}3&1&0\\-2&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}\frac{g+h}{2}&\frac{(g-h)^2}{4}&0\\1&\frac{g+h}{2}&0\\0&0&1\end{pmatrix},$$

respectively. □

4. A Result of Bernstein

In [1] Bernstein showed that the only zeros of the integer function

$$f(n) := \sum_{j\geq 0}(-1)^j\binom{n-2j}{j}$$

are at $n=3$ and $n=12$. We use Corollary 1 to relate the zeros of this function to solutions of a certain cubic Thue equation and hence to derive Bernstein's result.

Let

$$A = \begin{pmatrix}1&1&0\\0&0&1\\-1&0&0\end{pmatrix}.$$

With the notation of Corollary 1, $t=1$, $s=0$, $d=-1$, so that

$$a_n = \sum_{3j\leq n}(-1)^j\binom{n-2j}{j} = f(n),$$

and, for $n \geq 4$,

$$A^n = f(n-2)A^2 + (f(n) - f(n-2))A + (f(n) - f(n-1))\, I$$
$$= \begin{pmatrix} f(n) & f(n-1) & f(n-2) \\ -f(n-2) & -f(n-3) & -f(n-4) \\ -f(n-1) & -f(n-2) & -f(n-3) \end{pmatrix}.$$

The last equality follows from the fact that $f(k+1) = f(k) - f(k-2)$, for $k \geq 2$.

Now suppose $f(n-2) = 0$. Since the recurrence relation above gives that $f(n-4) = -f(n-1)$ and $f(n) = f(n-1) - f(n-3)$, it follows that

$$(-1)^n = \det(A^n) = \begin{vmatrix} f(n-1) - f(n-3) & f(n-1) & 0 \\ 0 & -f(n-3) & f(n-1) \\ -f(n-1) & 0 & -f(n-3) \end{vmatrix}$$

$$= -f(n-1)^3 - f(n-3)^3 + f(n-1)f(n-3)^2.$$

Thus $(x, y) = \pm(f(n-1), f(n-3))$ is a solution of the Thue equation

$$x^3 + y^3 - x\,y^2 = 1.$$

One could solve this equation in the usual manner of finding bounds on powers of fundamental units in the cubic number field defined by the equation $x^3 - x + 1 = 0$. Alternatively, the Thue equation solver in PARI/GP [3] gives unconditionally (in less than a second) that the only solutions to this equation are

$$(x, y) \in \{(4, -3), (-1, 1), (1, 0), (0, 1), (1, 1)\},$$

leading to Bernstein's result once again.

References

[1] Bernstein, Leon *Zeros of the functions* $f(n) = \sum_{i=0}(-1)^i\binom{n-2i}{i}$. J. Number Theory **6** (1974), 264–270.

[2] James Mc Laughlin, *Combinatorial Identities Deriving from the n-th Power of a* 2×2 *Matrix.* Integers **4** (2004), A19, 14 pp.

[3] PARI/GP: http://pari.math.u-bordeaux.fr/

[4] B. Sury, *A curious polynomial identity.* Nieuw Arch. Wisk. (4) **11** (1993), no. 2, 93–96.

[5] —–, *A parent of Binet's forumla?* Math. Magazine **77** (2004), no. 4, 308-310.

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