SYMMETRY AND SPECIALIZABILITY IN THE CONTINUED FRACTION EXPANSIONS OF SOME INFINITE PRODUCTS

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Abstract. Let $f(x) \in \mathbb{Z}[x]$. Set $f_0(x) = x$ and, for $n \geq 1$, define $f_n(x) = f(f_{n-1}(x))$.

We describe several infinite families of polynomials for which the infinite product

$$\prod_{n=0}^{\infty}\left(1+\frac{1}{f_n(x)}\right)$$

has a *specializable* continued fraction expansion of the form

$$S_\infty = [1; a_1(x),\, a_2(x),\, a_3(x), \ldots],$$

where $a_i(x) \in \mathbb{Z}[x]$ for $i \geq 1$.

When the infinite product and the continued fraction are *specialized* by letting x take integral values, we get infinite classes of real numbers whose regular continued fraction expansion is predictable.

We also show that, under some simple conditions, all the real numbers produced by this specialization are transcendental.

We also show, for any integer $k \geq 2$, that there are classes of polynomials $f(x,k)$ for which the regular continued fraction expansion of the product

$$\prod_{n=0}^{k}\left(1+\frac{1}{f_n(x,k)}\right)$$

is specializable but the regular continued fraction expansion of

$$\prod_{n=0}^{k+1}\left(1+\frac{1}{f_n(x,k)}\right)$$

is not specializable.

1. Introduction

The problem of finding the regular continued fraction expansion of an irrational quantity expressed in some other form has a long history but until the 1970's not many examples of such continued fraction expansions were known. Apart from the quadratic irrationals and numbers like e^q, for certain rational q, there were very few examples of irrational numbers with predictable patterns in their sequence of partial quotients.

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Being able to predict a pattern in the regular continued fraction expansion of an irrational number is not only interesting in its own right, but if one can also derive sufficient information about the convergents, it is then sometimes possible to prove that the number is transcendental.

In [10], Lehmer showed that certain quotients of modified Bessel functions evaluated at various rationals had continued fraction expansions in which the partial quotients lay in arithmetic progressions. He also showed that similar quotients of modified Bessel functions evaluated at the square root of a positive integer had continued fraction expansions in which the sequence of partial quotients consisted of interlaced arithmetic progressions.

An old result, originally due to Böhmer [3] and Mahler [11], was rediscovered by Davison [7] and Adams and Davison [1] (generalizing Davison's previous result in [7]). In this latter paper, the authors were able to determine, for any positive integer $a \geq 2$ and any positive irrational number α, the regular continued fraction expansion of the number

$$S_a(\alpha) = (a-1)\sum_{r=1}^{\infty}\frac{1}{a^{\lfloor r\alpha\rfloor}} \tag{1.1}$$

in terms of the convergents in the continued fraction expansion of α^{-1}. They were further able to show that all such numbers $S_a(\alpha)$ are transcendental.

A generalization of Davison's result from [7] was given by Bowman in [5] and Borwein and Borwein [4] gave a two-variable generalization of (1.1) but the continued fraction expansion in this latter case is not usually regular.

Shallit [15] and Kmošek [8] showed independently that the continued fraction expansions of the irrational numbers

$$\sum_{k=0}^{\infty}\frac{1}{u^{2^k}}$$

have predictable continued fraction expansions. This result was subsequently generalized by Köhler [9], by Pethö [13] and by Shallit [16] once again.

In [12], Mendès France and van der Poorten considered infinite products of the form

$$\prod_{h=0}^{\infty}\left(1+X^{-\lambda_h}\right),$$

where $0 < \lambda_1 < \lambda_2 < \cdots$ is any sequence of rational integers satisfying a certain growth condition and showed that such products had a predictable continued fraction expansion in which all the partial quotients were polynomials in $\mathbb{Z}[X]$. They further showed that if the infinite product and continued fraction were *specialized* by letting X be any integer $g \geq 2$, that all such real numbers

$$\gamma = \prod_{h=0}^{\infty}\left(1+g^{-\lambda_h}\right)$$

so obtained were transcendental. Similar investigations, in which the continued fraction expansions of certain formal Laurent series are determined, can be found in [19], [18], [20] and [2].

Let $f(x) \in \mathbb{Z}[x]$, $f_0(x) = x$ and, for $i \geq 1$, $f_i(x) = f(f_{i-1}(x))$, the i-th iterate of $f(x)$. In [17], Tamura investigated infinite series of the form

$$\theta(x : f) = \sum_{m=0}^{\infty} \frac{1}{f_0(x) f_1(x) \cdots f_m(x)}.$$

He showed, for all polynomials in a certain congruence class, that the continued fraction expansion of $\theta(x : f)$ had all partial quotients in $\mathbb{Z}[x]$. He further showed that if the series and continued fraction were specialized to a sufficiently large integer (depending on $f(x)$), then the resulting number was transcendental.

The infinite series $\sum_{k=0}^{\infty} 1/x^{2^k}$, investigated by Shallit [15] and Kmošek [8] may be regarded as a special case of the infinite series $\sum_{k=0}^{\infty} 1/f_k(x)$, with $f(x) = x^2$. In a very interesting paper, [6], Cohn gave a complete classification of all those polynomials $f(x) \in \mathbb{Z}[x]$ for which the series $\sum_{k=0}^{\infty} 1/f_k(x)$ had a continued fraction expansion in which all partial quotients were in $\mathbb{Z}[x]$. By then letting x take integral values, he was able to derive expansions such as the following:

$$\sum_{n\geq 0} \frac{1}{T_{4^n}(2)} = [0; 1, 1, 23, 1, 2, 1, 18815, 3, 1, 23, 3, 1, 23, 1, 2, 1,$$
$$106597754640383, 3, 1, 23, 1, 3, 23, 1, 3, 18815, 1, 2, 1, 23, 3, 1, 23, \cdots],$$

where $T_l(x)$ denotes the l-th Chebyshev polynomial, and also to derive the continued fraction expansion for certain sums of series.

At the end of Cohn's paper he listed a number of open questions and conjectures. One of the problems he mentioned was finding a similar classification of all those polynomials $f(x) \in \mathbb{Z}[x]$ for which the regular continued fraction expansion of the infinite product

$$\prod_{k=0}^{\infty} \left(1 + \frac{1}{f_k(x)}\right) \tag{1.2}$$

has all partial quotients in $\mathbb{Z}[x]$.

This turns out to be a technically more difficult problem. One reason is that, given any positive integer k, there are classes of polynomials such as $f(x,k) = 2x + x^2 + x^k((-1)^k + (1+x)g(x))$ for which the regular continued fraction expansion of the product $\prod_{n=0}^{k}(1 + 1/f_n(x,k))$ is specializable for all polynomials $g(x) \not\equiv (-1)^{k+1} (\text{mod } x)$ but the regular continued fraction expansion of $\prod_{n=0}^{k+1}(1 + 1/f_n(x,k))$ is not specializable. This is in contrast to the infinite series case dealt with by Cohn, where $\sum_{k=0}^{\infty} 1/f_k(x)$ had a specializable continued fraction expansion if and only if $\sum_{k=0}^{3} 1/f_k(x)$ had a specializable continued fraction expansion.

In this paper we give several infinite classes of polynomials for which $\prod_{n=0}^{\infty}(1+1/f_n(x))$ has a specializable regular continued fraction.

For the polynomials in these classes of degree at least three, we specialize the product at (1.2) by letting x take positive integral values, producing certain classes of real numbers. We examine the corresponding regular continued fractions to prove the transcendence of these numbers.

2. Some Preliminary Lemmas

Unless otherwise stated $f(x)$, $G(x)$, $g(x)$ will denote polynomials in $\mathbb{Z}[x]$, $f_0(x) := x$ and, for $n \geq 0$, $f_{n+1}(x) := f(f_n(x))$. Sometimes, for clarity and if there is no danger of ambiguity, $f(x)$ will be written as f and $f_n(x)$ as f_n. Likewise, $(f(x))^m$ will be written as f^m, $(f_n(x))^m$ as f_n^m, etc.

For a fixed $f(x) \in \mathbb{Z}[x]$, set

$$\prod_n(f(x)) = \prod_n(f) = \prod_n := \prod_{i=0}^{n}\left(1+\frac{1}{f_i}\right)$$

and

$$\prod_\infty(f(x)) = \prod_\infty(f) = \prod_\infty := \prod_{i=0}^{\infty}\left(1+\frac{1}{f_i}\right).$$

Similarly, $S_n(f(x)) = S_n(f) = S_n$ will denote the regular continued fraction expansion (via the Euclidean algorithm) of $\prod_n$ and $S_\infty(f(x)) = S_\infty(f) = S_\infty$ will denote the regular continued fraction expansion of $\prod_\infty$. (The more concise forms will be used when there is no danger of ambiguity.)

Unless stated otherwise, the sequence of partial quotients in S_n will be denoted by $\vec{w}_n$, so that $S_n = [\vec{w}_n]$.

If a partial quotient in a continued fraction is a polynomial in $\mathbb{Z}[x]$, it is said to be *specializable*. A continued fraction all of whose partial quotients are specializable is also called specializable. We say that a continued fraction $[a_0, a_1, \ldots, a_n]$ has *even* (resp. *odd*) length if n is even (resp. odd).

Since a form of the folding lemma will be used later, we state and prove this for the sake of completeness. In what follows let $\vec{w}$ denote the word $a_1, \ldots, a_n$, $\overleftarrow{w}$ the word $a_n, \ldots, a_1$ and $-\overleftarrow{w}$ the word $-a_n, \ldots, -a_1$. For $i \geq 0$, let A_i/B_i denote the i-th convergent of the continued fraction $[a_0, a_1, \ldots]$.

Recall that

$$\begin{aligned} A_{n+1} &= a_{n+1}A_n + A_{n-1}, \\ B_{n+1} &= a_{n+1}B_n + B_{n-1}, \end{aligned} \tag{2.1}$$

and

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}. \tag{2.2}$$

We need the following preliminary results.

Lemma 1. *For $j = 0, 1$,*

$$[(-1)^j \overleftarrow{w}] = (-1)^j \frac{B_n}{B_{n-1}}. \tag{2.3}$$

If $a_0 = 1$, then

$$[(-1)^j \vec{w}] = (-1)^j \frac{B_n}{A_n - B_n}, \tag{2.4}$$

and

$$[(-1)^j \overleftarrow{w}, (-1)^j] = (-1)^j \frac{A_n}{A_{n-1}}. \tag{2.5}$$

Proof. All of these follow easily from the correspondence between matrices and continued fractions (easily proved by induction or see [22]):

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} -a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -a_n & 1 \\ 1 & 0 \end{pmatrix} = (-1)^n \begin{pmatrix} -A_n & A_{n-1} \\ B_n & -B_{n-1} \end{pmatrix}.$$

□

Lemma 2. [19]

$$[a_0; \vec{w}, Y, -\overleftarrow{w}] = \frac{A_n}{B_n}\left(1 + \frac{(-1)^n}{Y A_n B_n}\right).$$

Proof. If we use (2.3), followed by (2.1) and then (2.2), we get that

$$\begin{aligned}
[a_0; \vec{w}, Y, -\overleftarrow{w}] &= [a_0, \vec{w}, Y, -B_n/B_{n-1}] \\
&= [a_0; \vec{w}, Y - B_{n-1}/B_n] \\
&= \frac{A_n(Y - B_{n-1}/B_n) + A_{n-1}}{B_n(Y - B_{n-1}/B_n) + B_{n-1}} \\
&= \frac{A_n}{B_n}\left(1 + \frac{(-1)^n}{Y A_n B_n}\right).
\end{aligned}$$

□

There are other forms of symmetry which will appear later so we give the lemma below. Note that in all of these cases $a_0 = 1$. We call these symmetries "doubling" symmetries, following Cohn [6].

Lemma 3.

$$[1; \vec{w}, Y, -\vec{w}] = \frac{A_n}{B_n}\left(1 + \frac{(-1)^n}{A_n(B_n(Y+1) - A_n + B_{n-1})}\right). \tag{2.6}$$

$$[1; \vec{w}, Y, -\overleftarrow{w}, -1] = \frac{A_n}{B_n}\left(1 + \frac{1}{(-1)^n Y A_n B_n - 1}\right). \tag{2.7}$$

$$[1;\vec{w},Y,\overleftarrow{w},1]=\frac{A_n}{B_n}\left(1+\frac{1}{(-1)^nB_n(YA_n+2A_{n-1})-1}\right). \tag{2.8}$$

$$[1;\vec{w},Y,\vec{w}]=\frac{A_n}{B_n}\left(1+\frac{(-1)^n}{A_n(B_n(Y-1)+A_n+B_{n-1})}\right). \tag{2.9}$$

Proof. We give the proof only for (2.6), as (2.7), (2.8) and (2.9) follow similarly. We use (2.4), followed by (2.1), to get that

$$\begin{aligned}[1;\vec{w},Y,-\vec{w}]&=[1;\vec{w},Y,-\frac{B_n}{A_n-B_n}]\\&=[1;\vec{w},Y+1-\frac{A_n}{B_n}]\\&=\frac{A_n\left(Y+1-\frac{A_n}{B_n}\right)+A_{n-1}}{B_n\left(Y+1-\frac{A_n}{B_n}\right)+B_{n-1}}.\end{aligned}$$

The result follows from (2.2), after some simple algebraic manipulation.

□

Cohn proved a version of (2.8) in [6]. We also point out that the doubling symmetry described at (2.6) occurs with some classes of polynomials such as the $f(x,k)=2x+x^2+x^k((-1)^k+(1+x)g(x))$ mentioned above. However S_n is not specializable for these polynomials, for $n\geq k+1$ (see Proposition 1) and we have not found S_∞ to be specializable for any polynomials that exhibit this kind of doubling symmetry.

For future reference we show how the various forms of symmetry found in the above lemma will be used. Suppose that $\prod_m$, when expanded as a continued fraction, is equal to $S_m=[1;\vec{w}]$, that the numerator of the ultimate convergent of S_m is A_m and the denominator of the ultimate convergent is B_m and that A'_m and B'_m are the numerator and denominator, respectively, of the penultimate convergent, that S_m is specializable and that S_{m+1} is related to S_m in one of the ways shown in Lemma 2 or Lemma 3. (Y_m is used here instead of Y to show the dependence on m). Then

$$\prod_{m+1}=\prod_m\left(1+\frac{1}{f_{m+1}}\right)=\frac{A_m}{B_m}\left(1+\frac{1}{f_{m+1}}\right).$$

On the other hand, from the above lemmas,

$$S_{m+1}=\frac{A_m}{B_m}\left(1+\frac{1}{H(A_m,B_m,A'_m,B'_m,Y_m)}\right),$$

where $H(A_m,B_m,A'_m,B'_m,Y_m)$ is a polynomial in its variables with integral coefficients that is linear in Y_m.

If solving the equation $f_{m+1}=H(A_m,B_m,A'_m,B'_m,Y_m)$ for Y_m leaves Y_m in $\mathbb{Z}[x]$ for all m then S_m is specializable for all m.

For later use we also note that if $x \mid (f+1)$ then $\prod_m$ simplifies to leave f_m in the denominator and, say, r_m in the numerator. If $(f_m, r_m) = 1$ then, up to sign, the final numerator convergent of S_m is r_m and the final denominator convergent is f_m. A similar situation also holds if $(x+1) \mid f$.

As a result of the following lemma, polynomials of degree 2 and those of degree 3 or more will be considered separately.

Lemma 4. *If $f(x)$ has degree greater than 2, then S_{n+1} contains S_n at the beginning of the expansion.*

Proof. Suppose $S_n = [1; a_1, \cdots, a_m] = p/q$ where the a_i's, p and q are polynomials in $\mathbb{Q}[x]$. Let $[1; a_1, \cdots, a_i] =: p_i/q_i$ and suppose that, via the Euclidean algorithm, we have that

$$\begin{aligned} p &= q + r_1. \\ q &= a_1 r_1 + r_2. \\ r_1 &= a_2 r_2 + r_3. \\ &\vdots \\ r_{m-2} &= a_{m-1} r_{m-1} + r_m. \\ r_{m-1} &= a_m r_m. \end{aligned} \tag{2.10}$$

By definition $\prod_{n+1} = p/q\,(1 + 1/f_{n+1}) = p(f_{n+1}+1)/(q f_{n+1})$ and to develop the continued fraction expansion of $\prod_{n+1}$ one can apply the Euclidean algorithm to this quotient. From (2.10):

$$\begin{aligned} p(f_{n+1}+1) &= q\, f_{n+1} + (r_1 f_{n+1} + p) \\ q f_{n+1} &= a_1(r_1 f_{n+1} + p) + (r_2 f_{n+1} - a_1 p) \\ (r_1 f_{n+1} + p) &= a_2(r_2 f_{n+1} - a_1 p) + (r_3 f_{n+1} + p(1 + a_1 a_2)) \\ &\ddots \end{aligned}$$

Let $r'_{-1} = p(f_{n+1}+1)$, $r'_0 = q f_{n+1}$ and for $1 \le i \le m$, set

$$r'_i = r_i f_{n+1} + (-1)^{i+1} p\, q_{i-1}.$$

We next show that, for $0 \le i \le m-1$,

$$r'_{i-1} = a_i r'_i + r'_{i+1}. \tag{2.11}$$

This is clearly true for $i = 0, 1$ ($a_0 = 1$). From (2.10), $r_{i+1} = r_{i-1} - a_i r_i$ and from the recurrence relation for the q_i's, $q_{i+1} = a_{i+1} q_i + q_{i-1}$. Suppose (2.11) is true for $i = 0, 1, \ldots, j-1$.

$$\begin{aligned} r'_{j-1} - a_j r'_j &= \left(r_{j-1} f_{n+1} + (-1)^j p\, q_{j-2}\right) - a_j \left(r_j f_{n+1} + (-1)^{j+1} p\, q_{j-1}\right). \\ &= (r_{j-1} - a_j r_j) f_{n+1} + (-1)^{j+2} p (q_{j-2} + a_j q_{j-1}). \\ &= r_{j+1} f_{n+1} + (-1)^{j+2} p\, q_j. \\ &= r'_{j+1}. \end{aligned}$$

Thus (2.11) is true for $0 \leq i \leq m-1$.

All that remains to prove the lemma is to show that the degree of r'_{i+1} is less than the degree of r'_i for $0 \leq i \leq m-1$.

Let the degree of a polynomial b be denoted by $\deg(b)$. From the Euclidean algorithm it follows that $\deg(r_{i+1}) < \deg(r_i)$. Suppose $\deg(f) = r \geq 3$ so that f_i has degree r^i and, since $\prod_{i=0}^{n}(1+1/f_i(x)) = p/q$, that

$$\deg(p),\, \deg(q) \leq 1 + r + r^2 + \cdots r^n = (r^{n+1}-1)/(r-1).$$

Thus, for $0 \leq i \leq m$,

$$\deg(p\, q_i) \leq \deg(p\, q) \leq 2(r^{n+1}-1)/(r-1) < r^{n+1} = \deg(f_{n+1}),$$

since $r \geq 3$. This implies that, for $0 \leq i \leq m-1$,

$$\begin{aligned}\deg(r'_{i+1}) &= \deg(r_{i+1}f_{n+1} + (-1)^{i+2}p\, q_i) = \deg(r_{i+1}f_{n+1}) \\ &< \deg(r_i f_{n+1}) = \deg(r_i f_{n+1} + (-1)^{i+1}p\, q_{i-1}) = \deg(r'_i).\end{aligned}$$

The result follows. □

Note that if $\deg(f) = 2$ (so that $\deg(f_j) = 2^j$) then the situation can be quite different.

Lemma 5. *Let $f(x)$ be a polynomial of degree two and suppose S_n begins with $[1; a_1, \ldots, a_k, a_{k+1}]$. If*

$$\deg(a_{k+1}) + 2\sum_{i=1}^{k}\deg(a_i) < 2^{n+1}, \tag{2.12}$$

then S_{n+1} begins with $[1; a_1, \ldots, a_k]$.

Proof. With the notation of Lemma 4 and its proof, $[1; a_1, a_2, \ldots, a_k]$ will be part of S_{n+1} if

$$\deg(r'_{i+1}) < \deg(r'_i), \qquad 0 \leq i \leq k. \tag{2.13}$$

Recall that $r'_{i+1} = r_{i+1}f_{n+1} + (-1)^{i+2}p\, q_i$, so that (2.13) will follow if

$$\deg(p\, q_i) < \deg(r_{i+1}f_{n+1}), \qquad 0 \leq i \leq k.$$

Let $0 \leq i \leq k$. Since $[1; a_1, \cdots, a_i] = p_i/q_i$, we have that

$$\deg(q_i) = \sum_{j=1}^{i}\deg(a_j). \tag{2.14}$$

It is clear from (2.10) that $\deg(r_j) = \deg(a_{j+1}) + \deg(r_{j+1})$. This implies that

$$\deg(r_{i+1}) = \deg(q) - \sum_{j=1}^{i+1}\deg(a_j). \tag{2.15}$$

Now (2.12), (2.14) and (2.15) imply that

$$\deg(q_i) + \deg(q) - \deg(r_{i+1}) < 2^{n+1} = \deg(f_{n+1}).$$

The result follows, since $\deg(p) = \deg(q)$. □

We return to the case $\deg(f) \geq 3$. The implication of Lemmas 4 and 5 is that if $\deg(f) \geq 2$, then it makes sense to talk of the continued fraction expansion of $\prod_{i=0}^{\infty}(1+1/f_i)$ and, furthermore, that if $\deg(f) \geq 3$, then S_∞ is a specializable continued fraction if and only if S_n is a specializable continued fraction for each integer $n \geq 0$.

Remark: At this stage we are not concerned with whether the polynomials which are the partial quotients in S_∞ have negative leading coefficients or take non-positive values for certain positive integral x. Negatives and zeroes are easily removed from regular continued fraction expansions (see [21], for example).

The following lemma means that we get the proof of the specializability of the regular continued fraction expansion of $\prod_{k=0}^{\infty}(1+1/f_k(x))$ for some classes of polynomials $f(x)$ for free.

Lemma 6. *Suppose $S_\infty(f)$ is specializable. Define $g(x)$ by*

$$g(x) = -f(-x-1) - 1. \tag{2.16}$$

Then $S_\infty(g)$ is specializable.

Proof. If $\prod_{k=0}^{\infty}(1+1/f_k(x))$ has a specializable continued fraction expansion $S_\infty(f(x)) := [1; a_1(x), a_2(x), \dots]$, then $\prod_{k=0}^{\infty}(1+1/f_k(-x-1))$ has the specializable continued fraction expansion

$$S_\infty(f(-x-1)) = [1; a_1(-x-1), a_2(-x-1), \dots].$$

Let $g(x)$ be defined as in the statement of the lemma. For $k \geq 0$,

$$g_k(x) = -f_k(-x-1) - 1.$$

This is clearly true for $k = 0, 1$. Suppose it is true for $k = 0, 1, \dots, m$.

$$\begin{aligned} g_{m+1}(x) &= g(g_m(x)) = g(-f_m(-x-1) - 1) \\ &= -f(-(-f_m(-x-1)-1)-1) - 1 = -f_{m+1}(-x-1) - 1. \end{aligned}$$

Next,

$$\begin{aligned} \prod_{\infty}(g(x)) &= \prod_{k=0}^{\infty}\left(\frac{1+g_k(x)}{g_k(x)}\right) = \prod_{k=0}^{\infty}\left(\frac{-f_k(-x-1)}{-f_k(-x-1)-1}\right) \\ &= \prod_{k=0}^{\infty}\left(\frac{f_k(-x-1)}{f_k(-x-1)+1}\right). \end{aligned}$$

From what has been said above, the final product has the regular continued fraction expansion $[0; 1, a_1(-x-1), a_2(-x-1), \dots]$ and is thus specializable. $\square$

We next demonstrate one of the difficulties in trying to arrive at a complete classification of all polynomials $f(x)$ for which $S_\infty(f)$ is specializable. We need the following lemmas.

Lemma 7. *Let k be an indeterminate and let t be a non-negative integer. Then*

$$(2.17)\quad (1+k)\sum_{m=0}^{t}(-1)^m(2k+k^2)^m = k^{t+2}h_t(k)+(-1)^t\left(2^{t+1}-1\right)k^{t+1}+\sum_{m=0}^{t}(-1)^m k^m,$$

where $h_t(k)\in\mathbb{Z}[k]$.

Proof. Upon taking the last term on the right side of (2.17) to the left side and simplifying, we get that

$$\begin{aligned}(1+k)&\sum_{m=0}^{t}(-1)^m(2k+k^2)^m-\sum_{m=0}^{t}(-1)^m k^m\\ &=(1+k)\frac{1-\left[-(2k+k^2)\right]^{t+1}}{1-\left[-(2k+k^2)\right]}-\frac{1-(-k)^{t+1}}{1-(-k)}\\ &=\frac{(-k)^{t+1}-\left[-(2k+k^2)\right]^{t+1}}{1+k}\\ &=(-k)^{t+1}\frac{1-(2+k)^{t+1}}{1+k}.\end{aligned}$$

The final quotient is clearly a polynomial in k, with constant term $1-2^{t+1}$. The result now follows. □

Lemma 8. *Let $k\geq 2$ be an integer and let $g(x)\in\mathbb{Z}[x]$ be such that $g(x)$ is not the zero polynomial if $k=2$. Define*

$$f(x):=2x+x^2+x^k((-1)^k+(x+1)g(x)).$$

For $0\leq n\leq k$, let

$$B_n=x\prod_{j=1}^{n}\frac{f_j}{f_{j-1}+1}.$$

Then

$$(2.18)\quad \frac{f_n^n}{B_n}=P_n(x)+\frac{2^{n(n-1)/2}}{x},$$

for some $P_n(x)\in\mathbb{Z}[x]$.

Proof. Since $x(x+1)\,|\,f$, it follows that $B_i\,|\,f_i^{i+1}$, for $i\geq 0$. This, together with the definition of B_n, give that

$$\frac{f_n^n}{B_n}=\frac{f_n^n(f_n+1)}{B_{n-1}f_n}=\frac{f_n^n}{B_{n-1}}+\frac{f_n^{n-1}}{B_{n-1}}$$

From what has been said just above, the first term is in $\mathbb{Z}[x]$ and from the definition of $f(x)$ it follows that

$$\frac{f_n^{n-1}}{B_{n-1}} = r_n(x) + 2^{n-1}\frac{f_{n-1}^{n-1}}{B_{n-1}},$$

for some $r_n(x) \in \mathbb{Z}[x]$. Thus

$$\frac{f_n^{n}}{B_{n}} = s_n(x) + 2^{n-1}\frac{f_{n-1}^{n-1}}{B_{n-1}},$$

for some $s_n(x) \in \mathbb{Z}[x]$. The result follows upon iterating this last expression downwards, noting that $B_0 = x$. □

Proposition 1. *Let $k \geq 2$ be an integer and let $g(x) \in \mathbb{Z}[x]$ be such that $g(x)$ is not the zero polynomial if $k = 2$. Define*

$$f(x) = 2x + x^2 + x^k((-1)^k + (x+1)g(x)). \tag{2.19}$$

Then $S_n(f)$ is specializable for $n \leq k$. If $g(x) \not\equiv (-1)^{k+1}$ (mod x), then S_n is not specializable for $n > k$.

Proof. We will show that the doubling symmetry at (2.6) can be used to develop the continued fraction expansion of $\prod_n$, $1 \leq n \leq k$. More precisely we will show that if $S_n = [1; \vec{w}_n]$ for $0 \leq n \leq k-1$, with each partial quotient in S_n a polynomial in $\mathbb{Z}[x]$, then

$$S_{n+1} = [1; \vec{w}_n, Y_n, -\vec{w}_n],$$

for some $Y_n \in \mathbb{Z}[x]$. We will then show that S_{k+1} is not specializable unless $g(x) \equiv (-1)^{k+1} \pmod{x}$ (which would have the effect of replacing k by $k+1$ in the statement of the form of $f(x)$ above) and this, together with Lemma 4, will give the result.

Note first of all that $S_0 = [1; x]$ and $S_1 = [1; x, -f/(x(x+1)), -x]$, so that the doubling symmetry at (2.6) occurs with $Y_0 = -f/(x(x+1))$. Next, let $n \in \{0, \dots, k-1\}$ and suppose $S_n = [1; \vec{w}_n]$ is specializable. We also suppose that S_j was developed from S_{j-1} via the doubling symmetry at (2.6), for $1 \leq j \leq n$ (so that $\vec{w}_n$ has odd length). Let A_n/B_n denote the final approximant and A'_n/B'_n the penultimate approximant of S_n. We further assume that

$$A_n = f_n + 1, \qquad B_n = x\prod_{j=1}^{n}\frac{f_j}{f_{j-1}+1}. \tag{2.20}$$

Note that this holds for $n = 0$, 1. We also assume that if $n \geq 1$, then

$$B_{n-1} \Bigg| \left(B'_{n-1} + \sum_{j=0}^{k-1}(-1)^{j+1}f_{n-1}^{j}\right). \tag{2.21}$$

This is true for $n = 1$ since $B_0 = x$, $B'_0 = 1$ and $f_0 = x$.

By the correspondence between continued fractions and matrices (see [22]),

$$[1;\vec{w}_n] \longleftrightarrow \begin{pmatrix} A_n & A'_n \\ B_n & B'_n \end{pmatrix}.$$

Further, from Lemma 1 and its proof,

$$\begin{aligned}
[1;\vec{w}_n, Y_n, -\vec{w}_n] &\longleftrightarrow \begin{pmatrix} A_n & A'_n \\ B_n & B'_n \end{pmatrix} \begin{pmatrix} Y_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_n & B'_n \\ A_n - B_n & B'_n - A'_n \end{pmatrix} \\
&= \begin{pmatrix} A_n^2 - A_n B_n(1+Y_n) - B_n A'_n & -A_n A'_n + A_n B'_n(1+Y_n) + A'_n B'_n \\ B_n(A_n - B_n - Y_n B_n - B'_n) & -B_n A'_n + B_n B'_n(1+Y_n) + B'^2_n \end{pmatrix} \\
&=: \begin{pmatrix} A_{n+1} & A'_{n+1} \\ B_{n+1} & B'_{n+1} \end{pmatrix}.
\end{aligned}$$

If we set

$$Y_n = -1 - \frac{-(1+f_n)^2 + f_{1+n} + B'_n(1+f_n)}{B_n\ (1+f_n)}, \tag{2.22}$$

and use the facts that $A_n = f_n + 1$ and that $\vec{w}_n$ has odd length (so that $A'_n = (-1 + B'_n A_n)/B_n = (-1 + B'_n(f_n+1))/B_n$, by the determinant formula), then we get

$$\begin{pmatrix} A_{n+1} & A'_{n+1} \\ B_{n+1} & B'_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + f_{1+n} & \dfrac{1 + f_n - B'_n - f_{1+n} B'_n}{B_n} \\ \dfrac{f_{1+n}}{1+f_n} B_n & 1 - \dfrac{f_{1+n}}{1+f_n} {B_n}' \end{pmatrix}. \tag{2.23}$$

It is clear that

$$\frac{A_{n+1}}{B_{n+1}} = \frac{(1+f_{1+n})(1+f_n)}{f_{1+n} B_n} = \frac{1+f_{1+n}}{f_{1+n}} \prod_n = \prod_{n+1},$$

so that $[1;\vec{w}_n, Y_n, -\vec{w}_n]$ gives the regular continued fraction expansion of $\prod_{n+1}$ and is specializable, provided $Y_n \in \mathbb{Z}[x]$. Note also that (2.20) now holds with n replaced by $n+1$.

We show $Y_n \in \mathbb{Z}[x]$. From the definition of $f(x)$ we have that

$$f_{n+1} = 2f_n + f_n^2 + f_n^k \left((-1)^k + (1+f_n) g(f_n) \right).$$

From (2.20) and the fact that $x(x+1) \mid f$, it follows that $B_n \mid f_n^{n+1}$, and since $0 \le n \le k-1$, $B_n \mid f_n^k$. Thus the result will follow if we can show that

$$B_n \left| \left(B'_n + \frac{(-f_n)^k - 1}{f_n + 1} \right) \right. \text{ or } B_n \left| \left(B'_n + \sum_{j=0}^{k-1} (-1)^{j+1} f_n^j \right) \right. . \tag{2.24}$$

Here and subsequently we mean divisibility in $\mathbb{Z}[x]$.

We now use the facts (clear from (2.23)) that

$$B_n = B_{n-1} \frac{f_n}{f_{n-1} + 1} \qquad \text{and} \qquad B'_n = 1 - B'_{n-1} \frac{f_n}{f_{n-1} + 1}$$

to get that (2.24) will follow if

$$B_{n-1}\frac{f_n}{f_{n-1}+1}\Bigg|\left(-B'_{n-1}\frac{f_n}{f_{n-1}+1}-f_n\sum_{j=0}^{k-2}(-1)^{j+1}f_n^j\right),$$

or

$$B_{n-1}\Bigg|\left(B'_{n-1}+\sum_{j=0}^{k-2}(-1)^{j+1}f_n^j(1+f_{n-1})\right). \tag{2.25}$$

By the same argument as that just before (2.24), it follows that $B_{n-1}\mid f_{n-1}^k$, so that (2.25) will hold if

$$B_{n-1}\Bigg|\left(B'_{n-1}+\sum_{j=0}^{k-2}(-1)^{j+1}(2f_{n-1}+f_{n-1}^2)^j(1+f_{n-1})\right). \tag{2.26}$$

By (2.17),

$$\sum_{j=0}^{k-2}(-1)^{j+1}(2f_{n-1}+f_{n-1}^2)^j(1+f_{n-1})=\sum_{j=0}^{k-2}(-1)^{j+1}f_{n-1}^j+f_{n-1}^{k-1}h(f_{n-1}),$$

with $h(z)\in\mathbb{Z}[z]$. Since $B_{n-1}\mid f_{n-1}^{k-1}$, we can ignore the second term on the right above and increase the index on the sum from $k-2$ to $k-1$ for free, and get that (2.26) will hold if

$$B_{n-1}\Bigg|\left(B'_{n-1}+\sum_{j=0}^{k-1}(-1)^{j+1}f_{n-1}^j\right). \tag{2.27}$$

However, this is true by (2.21) and thus $Y_n\in\mathbb{Z}[x]$. Note that (2.24) is (2.21) with n replaced by $n+1$, so that the induction can be continued and S_n is specializable for $0\le n\le k$.

We next show that if $g(x)=(-1)^{k+1}+b+x\,g_1(x)$, with $b\neq 0$ and $g_1(x)\in\mathbb{Z}[z]$, then S_{k+1} is not specializable. Define

$$Y'_k:=-1-\frac{-(1+f_k)^2+f_{1+k}+B'_k(1+f_k)}{B_k\ (1+f_k)}+\frac{2^{k(k-1)/2}b}{x}. \tag{2.28}$$

Firstly, we prove that $Y'_k\in\mathbb{Z}[x]$. If (2.19) is used to write f_{k+1} in terms of f_k and we recall that $B_k\mid f_k^{k+1}$, it can easily be seen that $Y'_k\in\mathbb{Z}[x]$ if it can be shown that

$$-\frac{-1+f_k^k\left[(-1)^k+(1+f_k)((-1)^{k+1}+b)\right]+B'_k(1+f_k)}{B_k\ (1+f_k)}+\frac{2^{k(k-1)/2}b}{x}\in\mathbb{Z}[x]. \tag{2.29}$$

The first fraction can be re-written as

$$-\left((-1)^{k+1}+b\right)\frac{f_k^k}{B_k}-\frac{\left(-1+(-f_k)^k\right)/(1+f_k)+B_k'}{B_k}. \tag{2.30}$$

By Lemma 8,

$$-\left((-1)^{k+1}+b\right)\frac{f_k^k}{B_k}=-\left((-1)^{k+1}+b\right)P_n(x)-\frac{2^{k(k-1)/2}\left((-1)^{k+1}+b\right)}{x}, \tag{2.31}$$

for some $P_n(x)\in\mathbb{Z}[x]$. The second term in (2.30) can be written as

$$\begin{aligned}
-\frac{-\sum_{j=0}^{k-1}(-f_k)^j+B_k'}{B_k}&=-\frac{-\sum_{j=0}^{k-1}(-f_k)^j+1-B_{k-1}'\dfrac{f_k}{1+f_{k-1}}}{B_{k-1}\frac{f_k}{1+f_{k-1}}}\\
&=\frac{-\sum_{j=0}^{k-2}(-f_k)^j(1+f_{k-1})+B_{k-1}'}{B_{k-1}}\\
&=s(x)+\frac{-\sum_{j=0}^{k-2}(-(2f_{k-1}+f_{k-1}^2))^j(1+f_{k-1})+B_{k-1}'}{B_{k-1}},
\end{aligned}$$

for some $s(x)\in\mathbb{Z}[x]$. Here we have used, in turn, the formulae from (2.23) relating B_k to B_{k-1} and B_k' to B_{k-1}', (2.19) to write f_k in terms of f_{k-1} and the fact that $B_{k-1}\,|\,f_{k-1}^k$. Next, we use Lemma 7 to get that

$$\begin{aligned}
&\frac{-\sum_{j=0}^{k-2}(-(2f_{k-1}+f_{k-1}^2))^j(1+f_{k-1})+B_{k-1}'}{B_{k-1}}\\
&=\frac{-f_{k-1}^k h_{k-2}(f_{k-1})+(-1)^{k-1}(2^{k-1}-1)f_{k-1}^{k-1}-\sum_{j=0}^{k-2}(-f_{k-1})^j+B_{k-1}'}{B_{k-1}}\\
&=t(x)+\frac{(-1)^{k-1}2^{k-1}f_{k-1}^{k-1}-\sum_{j=0}^{k-1}(-f_{k-1})^j+B_{k-1}'}{B_{k-1}}\\
&=t(x)+(-1)^{k-1}2^{k-1}\frac{f_{k-1}^{k-1}}{B_{k-1}}+\frac{-\sum_{j=0}^{k-1}(-f_{k-1})^j+B_{k-1}'}{B_{k-1}},
\end{aligned}$$

for some $t(x) \in \mathbb{Z}[x]$. Here again we have used the fact that $B_{k-1} \mid f_{k-1}^{k}$. Finally, Lemma 8 and (2.21) give that this last expression has the form

$$u(x) + \frac{(-1)^{k-1}2^{k(k-1)/2}}{x},$$

for some $u(x) \in \mathbb{Z}[x]$. Thus

$$-\frac{-\sum_{j=0}^{k-1}(-f_k)^j + B_k'}{B_k} = v(x) + \frac{(-1)^{k-1}2^{k(k-1)/2}}{x}, \tag{2.32}$$

for some $v(x) \in \mathbb{Z}[x]$. That $Y_k' \in \mathbb{Z}[x]$ now follows by (2.29), (2.30), (2.31) and (2.32).

Secondly, define α_k by

$$[1; \vec{w}_k, Y_k', \alpha_k] = \prod_{k+1} = \frac{A_k}{B_k}\left(1 + \frac{1}{f_{k+1}}\right).$$

Upon solving

$$\frac{\alpha_k(Y_k'A_k + A_k') + A_k}{\alpha_k(Y_k'B_k + B_k') + B_k} = \frac{A_k}{B_k}\left(1 + \frac{1}{f_{k+1}}\right)$$

for α_k and using (2.28) to eliminate Y_k' and the determinant formula to eliminate A_k', we find

$$\alpha_k = -\frac{B_k x}{2^{k(k-1)/2}bB_k + (1 + f_k - B_k)x}. \tag{2.33}$$

Since $A_k = 1 + f_k$ and $\prod_k = A_k/B_k$, f_k and B_k have the same degree and same leading coefficient, so that $(1+f_k-B_k)x$ has degree less than $B_k x$. This implies that α_k is a rational function whose numerator has higher degree in x than its denominator, so that S_{k+1} begins with $[1; \vec{w}_k, Y_k']$. Next,

$$\begin{aligned}(2.34)\quad &\left(\alpha_k - \frac{-x}{2^{k(k-1)/2}b + 1}\right)^{-1} \\ &\qquad = -\frac{\left(2^{k(k-1)/2}b + 1\right)\left(2^{k(k-1)/2}bB_k + (1 + f_k - B_k)x\right)}{x\left(B_k - x - f_k x + B_k x\right)}.\end{aligned}$$

If $b = 0$ then $f(x)$ has the form at (2.19), but with k replaced by $k+1$ and, from what has been shown already,

$$S_{k+1} = [1; \vec{w}_k, Y_k, -\vec{w}_k] = \left[1; \vec{w}_k, Y_k, -x, -\frac{1 + f_k - B_k}{B_k - x - f_k x + B_k x}\right].$$

The final term in the last continued fraction comes from letting $b = 0$ on the right side of (2.34) and is a rational function whose numerator has degree greater than its numerator. (This must be the case since when $b = 0$, S_{k+1} has the form $[1; \vec{w}_k, Y_k, -x, \ldots]$, as each $\vec{w}_k$ begins with x.) This implies that the rational function on the right side of (2.34) has the same property and so, when $b \neq 0$,

$$S_{k+1} = \left[1; \vec{w}_k, Y'_k, \frac{-x}{2^{k(k-1)/2}b+1}, \dots\right]$$

and is thus not specializable. The proof is now complete by Lemma 4. $\square$

Corollary 1. *Let $k \geq 2$ be an integer and let $g(x) \in \mathbb{Z}[x]$ be such that $g(x) \neq 0$ if $k = 2$. Let*

$$f(x) = -x^2 - (1+x)^k \left(1 + (-1)^{k+1}\, x\, g(x)\right).$$

Then S_n is specializable for $0 \leq n \leq k$. If $g(x) \not\equiv (-1)^{k+1} (mod\ (x+1))$, then S_n is not specializable for $n > k$.

Proof. This follows from Proposition 1 and Lemma 6. $\square$

One reason we proved Proposition 1 was to show that it is not possible to eliminate all classes of polynomials for which S_∞ is not specializable by simply looking at the continued fraction expansion of a finite number of terms of the infinite product for a general polynomial (Cohn was able to do this in the infinite series case by looking at just the first four terms).

3. Specializability of S_∞ for various infinite families of polynomials of degree greater than two

We can now show that the specializability of S_n occurs for all n for all polynomials in several infinite families. We have the following theorem.

Theorem 1. *Let $f(x)$, $G(x)$ and $g(x)$ denote non-zero polynomials in $\mathbb{Z}[x]$ such that the degree of $f(x)$ is at least three. If $f(x)$ has one of the following forms,*

$$\begin{aligned}
(i)\ & f(x) = x^2(x+1)g(x),\\
(ii)\ & f(x) = x(x+1)G(x) - x - 1,\\
(iii)\ & f(x) = x(x+1)^2 g(x) - 1\\
(iv)\ & f(x) = x(x^2-1)g(x) + 2x^2 - 1\\
(v)\ & f(x) = (x+1)(x(x+2)g(x) - 2(x+1))\\
(vi)\ & f(x) = x^2(x^2-1)g(x) + x^2,\\
(vii)\ & f(x) = x(x+1)((x+2)(x+1)g(x) - 1) - x - 2,
\end{aligned}$$

then, for each $n \geq 0$, S_n is a specializable continued fraction. Hence S_∞ is a specializable continued fraction.

Proof. We note that the proof of (iii) follows from the proof of (i) and Lemma 6 and that the proof of (v) likewise follows from the proof of (iv) and Lemma 6. However, we give independent proofs of (iii) and (v) since we also wish to demonstrate the types of doubling symmetry exhibited by the corresponding continued fractions. The proof of (vii) can similarly be deduced from the proof of (vi) and in this case no independent proof is given (doubling symmetry is not involved for cases (vi) and (vii)).

As in the proof of Proposition 1, throughout let A_i/B_i denote the final approximant, and A'_i/B'_i the penultimate approximant, of $S_i = [1; \vec{w}_i]$, for each $i \geq 0$.

(i) For this class of polynomials we will show that S_{m+1} is derived from S_m via the type of symmetry exhibited in the folding lemma (Lemma 2). $S_0 = [1; x]$ is clearly specializable. Suppose that S_m is specializable. From Lemma 2 and the discussion following Lemma 3 it is clear that S_{m+1} is specializable if $A_m B_m \mid f_{m+1}$ in $\mathbb{Z}[x]$. Since $f(x) = x^2(x+1)g(x)$ it follows that, for $i \geq 0$,

$$f_i^2(f_i + 1) \,|\, (f_{i+1} + 1). \tag{3.1}$$

Since $(x+1) \mid f$ we get after cancellation that

$$\prod_i = \frac{f_i + 1}{x \prod_{j=0}^{i-1} f_j^2 g(f_j)}.$$

Since $f_j | f_{j+1}$ for $j \geq 0$, each term in the denominator of the expression divides f_i and thus the numerator and denominator are relatively prime. Thus, up to sign, $A_i = f_i + 1$ and $B_i = f_{i-1}^2 g(f_{i-1}) B_{i-1}$. (The first of these holds for $i \geq 0$ and the second for $i \geq 1$). It follows easily by induction that $B_i \mid f_i^2$. The facts that $B_m \mid f_m^2$ and $A_m = \pm(f_m + 1)$ together with (3.1) give that

$$A_m B_m \mid f_{m+1}.$$

Hence the result.

(ii) For this class of polynomial it will be shown that S_m is derived from S_{m-1} by adding a single new partial quotient. It is clear from the definition of $f(x) = x(x+1)G(x) - x - 1$ that, for $i \geq 0$,

$$(f_i + 1)|f_{i+1}, \qquad f_i|(f_{i+1} + 1), \qquad f_i|f_{i+2}. \tag{3.2}$$

This implies that

$$\prod_i = \frac{f_i + 1}{x \prod_{j=0}^{i-1} (f_j G(f_j) - 1)} = \frac{(x+1) \prod_{j=0}^{i-1} ((f_j + 1)G(f_j) - 1)}{f_i}. \tag{3.3}$$

This gives that $A_i|(f_i + 1)$, and $B_i|f_i$, for all $i \geq 0$. Next,

$$\frac{A_{i+2}}{B_{i+2}} = \frac{A_i}{B_i} \frac{(f_{i+1} + 1)}{f_{i+1}} \frac{(f_{i+2} + 1)}{f_{i+2}} = \frac{A_i}{B_i} \frac{((f_{i+1} + 1)G(f_{i+1}) - 1)}{(f_{i+1} G(f_{i+1}) - 1)}$$

We next show that

$$(A_i, f_{i+1} G(f_{i+1}) - 1) = (B_i, (f_{i+1} + 1)G(f_{i+1}) - 1) = 1,$$

so that, up to sign,

$$A_{i+2} = ((f_{i+1}+1)G(f_{i+1})-1)A_i \tag{3.4}$$
$$B_{i+2} = (f_{i+1}G(f_{i+1})-1)B_i.$$

That $(B_i, (f_{i+1}+1)G(f_{i+1})-1) = 1$ is easily seen to be true since $B_i \mid f_i$, $f_i \mid f_{i+2}$, so that $B_i \mid f_{i+2}$, but $((f_{i+1}+1)G(f_{i+1})-1) \mid (f_{i+2}+1)$. The proof that $(A_i, f_{i+1}G(f_{i+1})-1) = 1$ is similar. We are now ready to prove that S_n is specializable for $n \geq 0$.

Initially, $S_0 = [1; x]$ and $S_1 = [1; x, -G]$. It will be shown by induction that $S_i = [1; \alpha_1, \dots, \alpha_{i+1}]$, where all the $\alpha_j' s \in \mathbb{Z}[x]$ and $(-1)^i f_i = A_{i-1}B_i$. Both statements are easily seen to be true for $i = 0, 1$.

Suppose these statements are true for $i = 0, 1, \dots, m-1$. Let $S_{m-1} = [1; \alpha_1, \dots, \alpha_m]$. Set

$$\alpha_{m+1} = -\frac{(f_{m-1}+1)}{A_{m-1}} G(f_{m-1})A_{m-2}, \tag{3.5}$$

which is in $\mathbb{Z}[x]$, since $A_{m-1} \mid (f_{m-1}+1)$, by the remark following (3.3). Let C_{m+1} be the numerator of the final convergent of $[1; \alpha_1, \dots, \alpha_m, \alpha_{m+1}]$ and let D_{m+1} be the denominator of the final convergent.

$$C_{m+1} = \alpha_{m+1}A_{m-1} + A_{m-2} = -((f_{m-1}+1)G(f_{m-1})-1)A_{m-2}$$
$$D_{m+1} = \alpha_{m+1}B_{m-1} + B_{m-2} = -(f_{m-1}G(f_{m-1})-1)B_{m-2}$$

The final equality for D_{m+1} uses the facts that $A_{m-1}B_{m-2} - A_{m-2}B_{m-1} = (-1)^{m-1}$ and $(-1)^{m-1}f_{m-1} = A_{m-2}B_{m-1}$. Hence, by (3.4), $C_{m+1}/D_{m+1} = A_m/B_m = \prod_m$ and $S_m = [1; \alpha_1, \dots, \alpha_m, \alpha_{m+1}]$. Finally,

$$\begin{aligned}
A_{m-1}B_m &= A_{m-1}(\alpha_{m+1}B_{m-1} + B_{m-2}) \\
&= -(f_{m-1}+1)G(f_{m-1})A_{m-2}B_{m-1} + A_{m-1}B_{m-2} \\
&= -(f_{m-1}+1)G(f_{m-1})(-1)^{m-1}f_{m-1} + (-1)^{m-1}f_{m-1} + (-1)^{m-1} \\
&= (-1)^m(f_{m-1}+1)(f_{m-1}G(f_{m-1})-1) = (-1)^m f_m.
\end{aligned}$$

The third equality also uses the facts that $A_{m-2}B_{m-1} = (-1)^{m-1}f_{m-1}$ and $A_{m-1}B_{m-2} - A_{m-2}B_{m-1} = (-1)^{m-1}$. Hence S_n is specializable for all n.

(iii) It will be shown that S_{m+1} is derived from S_m via the doubling symmetry found in (2.7). Suppose $S_m = [1; \vec{w_m}]$. It will be shown that Y_m can be chosen such that

$$S_{m+1} = [1; \vec{w_m}, Y_m, -\overleftarrow{w_m}, -1], \qquad Y_m \in \mathbb{Z}[x]. \tag{3.6}$$

Note that $S_0 = [1; x]$ and that $S_1 = [1; x, -G, -x, -1]$. S_1 has even length and if $S_2, \dots, S_m$ have been defined using (3.6), then S_m has even length. It can be seen from (2.7) that if $S_m = A_m/B_m$ and has even length, then $f_{m+1} = A_mB_mY_m - 1$ and $Y_m \in \mathbb{Z}[x]$ if $A_mB_m \mid (f_{m+1}+1)$. This we now show.

Since $f(x) = x(x+1)^2g(x) - 1$, it follows that $f_j \mid (f_{j+1}+1)$. After cancellation,

$$\prod_i = \frac{(x+1)\prod_{j=0}^{i-1}(f_j+1)^2 g(f_j)}{f_i},$$

so that $A_i \mid \left((x+1)\prod_{j=0}^{i-1}(f_j+1)^2 g(f_j)\right)$ and $B_i \mid f_i$. Thus it will be sufficient to show that

$$f_m(x+1)\prod_{j=0}^{m-1}(f_j+1)^2 g(f_j) \,\Big|\, (f_{m+1}+1).$$

Suppose that

$$f_i(x+1)\prod_{j=0}^{i-1}(f_j+1)^2 g(f_j) \,\Big|\, (f_{i+1}+1),$$

for $i = 0, 1, \ldots, m-1$ (this is clearly true for $i = 0$). Then

$$(x+1)\prod_{j=0}^{m-2}(f_j+1)^2 g(f_j) \,\Big|\, (f_m+1).$$

Since $(f_{m-1}+1)^2 g(f_{m-1}) \mid (f_m+1)$ it follows that

$$\Longrightarrow f_m(x+1)\prod_{j=0}^{m-1}(f_j+1)^2 g(f_j) \,\Big|\, f_m(1+f_m)^2.$$

This completes the proof of (iii), since $f_m(1+f_m)^2 \mid (f_{m+1}+1)$.

(iv) The argument is similar to that used in the proof of (iii). It will be shown that S_{m+1} is derived from S_m using the doubling symmetry found in (2.8).

Note that $S_1 = [1; x, -(x-1)g(x)-2, x, 1]$ and by induction we assume S_m has the symmetric form exhibited in (2.8), so that $A'_m = B_m$. Note also that the induction means that S_m has even length, since the duplicating formula always produces a continued fraction of even length.

It can be seen from (2.8) that

$$S_{m+1} = [1; \vec{w}_m, Y_m, \overleftarrow{w}_m, 1]$$

and will be specializable if the equation

$$B_m(A_m Y_m + 2A'_m) = f_{m+1} + 1 \tag{3.7}$$

is solvable with $Y_m \in \mathbb{Z}[x]$.

Since $f(x) = x(x^2-1)g(x) + 2x^2 - 1$ it can be seen that, for $i \geq 0$,

$$f_i \mid (f_{i+1}+1), \qquad (f_i^2 - 1) \mid (f_{i+1}-1). \tag{3.8}$$

After cancellation,

$$\prod_m = \frac{(x+1)\prod_{j=0}^{m-1}((f_j^2-1)g(f_j)+2f_j)}{f_m}. \tag{3.9}$$

Also, (3.8) implies that

$$\prod_{j=0}^{m}(1+f_j)\,\Big|\,(f_m^2-1),$$

so that the numerator and denominator in (3.9) above are relatively prime. Thus, up to sign $B_m = f_m$ and $A_m \,|\, (f_m^2-1)$.

Let

$$Y_m = \frac{f_m}{B_m}\frac{f_m^2-1}{A_m}g(f_m),$$

so that $Y_m \in \mathbb{Z}[x]$. Upon using the facts that $B_m = \pm f_m$ and (from above) $A'_m = B_m$, we get that

$$\begin{aligned}B_m(A_mY_m+2A'_m) &= B_mA_mY_m+2B_m^2\\ &= f_m(f_m^2-1)g(f_m)+2f_m^2\\ &= f_{m+1}+1.\end{aligned}$$

The result now follows by (3.7).

Cohn also gave a proof of (iv) in [6].

(v) In this case it will be shown that S_{m+1} is derived from S_m using the doubling symmetry found at (2.9).

Since $S_1 = [1; x, -G, x]$ and $\vec{w_i}$ symmetric implies $\vec{w_i}, Y_i, \vec{w_i}$ is symmetric, we have by induction that S_m has odd length and that $\vec{w}_m$ is symmetric. This gives that $B'_m = A_m - B_m$.

It can thus be seen from (2.9) that $[1; \vec{w_m}, Y_m, \vec{w_m}]$ will equal S_{m+1} and be specializable if the equation

$$f_{m+1} = -A_m(B_m(Y_m-2)+2A_m) \tag{3.10}$$

leads to $Y_m \in \mathbb{Z}[x]$.

Since $f(x) = (x+1)(x(x+2)g(x)-2(x+1))$ it follows that $(f_j+1)\,|\,f_{j+1}$.

After cancellation,

$$\prod_m = \frac{f_m+1}{x\prod_{j=0}^{m-1}(f_j(f_j+2)g(f_j)-2(f_j+1))}. \tag{3.11}$$

Further, since $x(x+2)\,|\,(f+2)$, it follows that

$$\prod_{j=0}^{m-1} f_j\,\Big|\,(f_m+2).$$

Thus the numerator and denominator in (3.11) above are relatively prime so that, up to sign, $A_m = f_m + 1$ and $B_m \mid f_m(f_m+2)$. Let

$$Y_m = 2 - \frac{f_m(f_m+1)(f_m+2)}{A_m B_m} g(f_m),$$

so that $Y_m \in \mathbb{Z}[x]$. The result now follows from (3.10), since

$$\begin{aligned} -A_m B_m (Y_m - 2) - 2A_m^2 &= (f_m+1) f_m (f_m+2) g(f_m) - 2(f_m+1)^2 \\ &= f_{m+1}. \end{aligned}$$

(vi) It will be shown that, for this class of polynomials and $m \geq 1$, S_{m+1} is derived from S_m by adding two terms. More precisely, if $m \geq 1$, $S_m = [1; x, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m]$ is specializable and

$$\begin{aligned} \alpha_{m+1} &:= -\frac{(f_{m+1} - f_m^2)}{A_m B_m}, \\ \beta_{m+1} &:= -A_m B_m, \end{aligned} \tag{3.12}$$

then $\alpha_{m+1}, \beta_{m+1} \in \mathbb{Z}[x]$ and $S_{m+1} = [1; x, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m, \alpha_{m+1}, \beta_{m+1}]$.

Initially, $S_0 = [1; x]$, $S_1 = [1; x, -xg(x)(x-1) - 1, -x(x+1)]$ and

$$S_2 = [1; x, -xg(x)(x-1) - 1, -x(x+1), -\frac{f(f-1)g(f)}{x(x+1)}, -x(x+1)f(f+1)],$$

so that (3.12) holds for $m = 1$. As part of the proof, it will be shown that, for $i \geq 1$,

$$A_i = \prod_{j=0}^{i} (f_j + 1), \qquad B_i = \prod_{j=0}^{i} f_j, \qquad A_i' = -\frac{f_i}{B_{i-1}}. \tag{3.13}$$

These equations are easily shown to be true for $i = 1$. Suppose that S_i has been defined via (3.12) for $i = 2, \dots, m$, that the conditions at (3.13) are true for $i = 1, \dots, m$ and that S_m is specializable.

We first show that $\alpha_{m+1} \in \mathbb{Z}[x]$ (clearly $\beta_{m+1} \in \mathbb{Z}[x]$ if S_m is specializable). Since $f = x^2((x^2-1)g(x)+1)$, we have that $x^2 \mid f$ and $(x^2-1) \mid (f-1)$, which imply that

$$\begin{aligned} \prod_{j=0}^{m-1} f_j \,&\Big|\, f_m, \\ \prod_{j=0}^{m-1} (f_j + 1) \,&\Big|\, (f_m - 1), \end{aligned}$$

These conditions with (3.13) imply that $A_m B_m \mid f_m^2(f_m^2 - 1)$ and hence that $A_m B_m \mid (f_{m+1} - f_m^2)$ and thus that $\alpha_{m+1} \in \mathbb{Z}[x]$.

Since $S_0 = [1; x]$, each S_i has odd length (in particular, S_m has odd length). Consider the following matrix product:

$$
\begin{aligned}
&\begin{pmatrix} A_m & A'_m \\ B_m & B'_m \end{pmatrix} \begin{pmatrix} \alpha_{m+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_{m+1} & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} A_m(\alpha_{m+1}\beta_{m+1}+1) + A'_m\beta_{m+1} & A_m\alpha_{m+1} + A'_m \\ B_m(\alpha_{m+1}\beta_{m+1}+1) + B'_m\beta_{m+1} & B_m\alpha_{m+1} + B'_m \end{pmatrix} \\
&\qquad = \begin{pmatrix} A_m(f_{m+1} - f_m^2 + 1) - A'_m A_m B_m & -\dfrac{f_{m+1} - f_m^2}{B_m} + A'_m \\ B_m(f_{m+1} - f_m^2 + 1) - B'_m A_m B_m & -\dfrac{f_{m+1} - f_m^2}{A_m} + B'_m \end{pmatrix} \\
&\qquad = \begin{pmatrix} A_m(f_{m+1}+1) & -\dfrac{f_{m+1}}{B_m} \\ B_m f_{m+1} & \dfrac{-f_{m+1}+1}{A_m} \end{pmatrix} \\
&\qquad =: \begin{pmatrix} C_{m+1} & C'_{m+1} \\ D_{m+1} & D'_{m+1} \end{pmatrix}.
\end{aligned}
$$

For the fourth equality we have used the facts (induction step) that $B_m = f_m B_{m-1}$, $A_m B'_m - A_m B'_m = 1$ (since S_m has odd length) and $A'_m = -f_m / B_{m-1}$. By the definition of C_{m+1}, D_{m+1},

$$\frac{C_{m+1}}{D_{m+1}} = \frac{A_m(1+f_{m+1})}{B_m f_{m+1}} = \prod_m \left(1 + \frac{1}{f_{m+1}}\right) = \prod_{m+1}.$$

Thus, from the relationship between matrices and continued fractions, we have that

$$S_{m+1} = [1; x, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m, \alpha_{m+1}, \beta_{m+1}]$$

and

$$\begin{pmatrix} A_{m+1} & A'_{m+1} \\ B_{m+1} & B'_{m+1} \end{pmatrix} = \begin{pmatrix} A_m(f_{m+1}+1) & -\dfrac{f_{m+1}}{B_m} \\ B_m f_{m+1} & \dfrac{-f_{m+1}+1}{A_m} \end{pmatrix}.$$

This equation also implies that (3.13) holds for $i = m+1$ and the result follows.

(vii) This follows from (vi) and Lemma 6. □

4. The Degree Two Case

In this section a complete classification is given of all polynomials $f(x)$ of degree two for which S_∞ is specializable or can be transformed in a simple way to produce a continued fraction which is specializable.

Essentially, the method is to start with a general polynomial

$$f(x) = ax^2 + (b-1)x + c - b - 1, \qquad a \neq 0,$$

(this form makes the continued fraction a little easier to work with) and to choose an integer n large enough so that some part of the continued fraction expansion of $\prod_n$, say $[1; a_1(x), \dots, a_t(x)]$, forms part of the continued fraction expansion of $\prod_\infty$ (This follows by Lemma 5). The coefficients in the $a_i(x)$ will be rational functions in a, b and c and the requirement that the $a_i(x) \in \mathbb{Z}[x]$, or that S_∞ can be transformed to produce a continued fraction that is specializable, will impose limiting conditions on a, b and c, leading to the stated classification.

Define

$$\begin{aligned}
num :=& (1 + b + a\,b - c - a\,c)\,\left(-1 + a^2 - 2\,a\,b + a\,c\right) + a\,(a-b)\,(b-c)\,x\\
&+ f\,\Big((1 + a - b + a\,b - a\,c)\,\left(-1 + a^2 - 2\,a\,b + a\,c\right) + a\,(a-b)^2\,x\Big)\\
&\qquad + (1 + a\,b - a\,c)\,\left(-1 + a^2 - 2\,a\,b + a\,c\right)\,f_2,\\
den :=& a\Bigg(\,(b-c)\,\left(1 - a^2 + 2\,a\,b - a\,c\right)\\
&\times\Big[\left(-1 + b - b^2 + a^2\,(-1 - b + c) + a\,\left(-1 + b + b^2 + c - b\,c\right)\right)\,f\\
&-1 - (-1 + a)\,b^2 - (-1 + a\,(-2 + c))\,c - b\,(1 - 2\,a\,(-1 + c) + c)\Big]\\
&+ (b - c + (a-b)\,f)\,x\\
&\quad\times\Big[-1 + b - b^2 + a^4\,(b-c) + a\,(-1 + b\,(-1 + 3\,b - 2\,c) + 2\,c)\\
&\qquad - a^3\,\left(-1 + b\,(-1 + 3\,b) + c - 4\,b\,c + c^2\right)\\
&\qquad\quad + a^2\,(2\,b - c)\,\left(-2 + b^2 + c - b\,(1 + c)\right)\Big]\Bigg),\\
\beta :=& -\frac{a\,(1 + a\,b - a\,c)^2\,\left(-1 + a^2 - 2\,a\,b + a\,c\right)^2 num}{(a-b)^4\,den}.
\end{aligned}$$

Then (preferably using a computer algebra system such as *Mathematica*) it can be shown that

$$\begin{aligned}
\prod_2 = \Bigg[1; -\frac{1}{a} + x, -\frac{a\,\left(-1 - b^2 + a\,c\right)}{(a-b)^2} - \frac{a^2\,x}{a-b},\\
\frac{(a-b)^2}{a\,(1 + a\,b - a\,c)^2\,(-1 + a^2 - 2\,a\,b + a\,c)^2} \times\\
\Big(-1 + b^3 + a^4 (b-c)^2 + a\,(-(b\,(4 + 3b)) + 3c) + a^2\,\left(1 - 5b^2 - 3c^2 + b\,(3 + 8c)\right)\\
+ a^3\,\left(-1 - 2\,b^3 - 2\,c + 5\,b^2\,c + c^3 + b\,\left(2 - 4\,c^2\right)\right)\Big)\\
+ \frac{(a-b)^3\,x}{(1 + a\,b - a\,c)\,(-1 + a^2 - 2\,a\,b + a\,c)}, \beta\Bigg].
\end{aligned} \tag{4.1}$$

In what follows we will make use of a remark of Cohn in [6]: that if the first partial quotient in a continued fraction with non-integral coefficients has a non-integral coefficient other than the constant term then the continued fraction is not specializable. (We will see that some continued fractions with partial quotients in which the constant term is non-integral can be transformed to make them specializable). Also, polynomials whose coefficients satisfy one of the conditions

$$(4.2)\qquad a-b=0, \qquad 1+a\,b-a\,c=0, \qquad -1+a^2-2\,a\,b+a\,c=0$$

will be considered separately. If none of these three equalities hold, then the numerator of β has degree four and the denominator has degree three. Note that the cofactor of $(b-c+(a-b)\,f)\,x$ in den is not zero for any triple of integers (a,b,c). This means that if the coefficients of $f(x)$ do not satisfy one of the conditions at (4.2), then the next regular partial quotient in S_2 is linear in x, so that

$$\deg(a_4(x))+2\sum_{i=1}^{3}\deg(a_i(x))=7<2^3.$$

Thus, by Lemma 5, S_n begins with the first four partial quotients in the continued fraction at (4.1), if $n\geq 2$.

For specializability, it is necessary to have $(b-a)\,|\,a^2$ in the third partial quotient (the case $a=b$ is to be examined separately). Write $b-a=u^2v$, with v square-free. Since $u^2\,|\,a^2$, then $u\,|\,a$, so write $a=us$. Since $u^2v\,|\,a^2$, then $v\,|\,s^2$, which implies $v\,|\,s$ (v is square-free), or $s=v\,w$. Thus, for specializability, it is necessary to have

$$a=u\,v\,w, \qquad\qquad b=u^2v+u\,v\,w,$$

for some integers u, v and w. If we substitute for a and b in the coefficient of x in the fourth partial quotient, then specializability requires

$$\frac{u^6\,v^3}{(-1+u\,v\,w\,\left(c-u\,v\,\left(u+w\right)\right))\,\left(-1+u\,v\,w\,\left(c-u\,v\,\left(2\,u+w\right)\right)\right)}\in\mathbb{Z}.$$

A check shows that happens only for

$$(a,b,c)\in\{(2,3,4),(-2,-3,-4),(2,1,1),(-2,-1,-1)\},$$

or

$$f\in\{2x^2+2x,\ -2x^2-4x-2,\ 2x^2-1,\ -2x^2-2x-1\}.$$

That $\prod_{\infty}$ is not specializable for the first and fourth polynomials follows from consideration of S_3 and Lemma 5. We will show that specializability occurs for the third polynomial and specializability for the second will follow from this fact and Lemma 6.

We next consider the case $a=b$, proceeding as previously. Suppose

$$f=ax^2+(a-1)x+c-a-1,$$

and we define

$$\begin{aligned} num &:= \left(1+a^2-a\,c\right)\,\big[\,(1+f)\,(1+x)\,(-1+a\,x)\,(1+f_3)+\\ &\quad f_2\,((1+f)\,(1+x)\,(-1+a\,x)-(f-(1+x)\,(-1+a\,x))\,f_3)\,\big],\\ den &:= a^2\,x\,(1+x)\,\big[-(a\,f\,f_2\,f_3)\\ &\quad +\left(-1+a\,\left(-1+c-x+a\,\left(-1+x+x^2\right)\right)\right)\\ &\qquad\qquad\qquad \times\,((1+f)\,(1+f_3)+f_2\,(1+f+f_3))\,\big],\\ \beta &:= \frac{num}{den}. \end{aligned}$$

Then (preferably once again using a computer algebra system such as *Mathematica*) it can be shown that

$$\prod_3 = \left[1;-\frac{1}{a}+x, a+\frac{a^3\,x}{-1-a^2+a\,c}+\frac{a^3\,x^2}{-1-a^2+a\,c}, \beta\right]. \tag{4.3}$$

Further, the numerator of β has degree twelve and the denominator has degree ten and the leading coefficient in the numerator or denominator does not vanish except in the case $\left(1+a^2-a\,c\right)$, which is examined separately. This all means that, apart from this exceptional case, the next partial quotient in the regular expansion of $\prod_3$ has degree two. Thus

$$\deg(a_3(x))+2\sum_{i=1}^{2}\deg(a_i(x))=8<2^4,$$

so that S_n starts with

$$\left[1;-\frac{1}{a}+x, a+\frac{a^3\,x}{-1-a^2+a\,c}+\frac{a^3\,x^2}{-1-a^2+a\,c},\dots\right]$$

for $n\geq 3$ (this once again by Lemma 5). This in turn implies that specializability requires

$$(-1-a^2+a\,c)\,|\,a^3,$$

and it is not difficult to see that this needs $-1-a^2+a\,c=\pm 1$. A check shows that the only solutions in this case are

$$(a,b,c)\in\{(a,a,a),(1,1,3),(-1,-1,-3),(2,2,3),(-2,-2,-3)\},$$

or

$$f\in\{ax^2+(a-1)x-1,\, x^2+1,\, -x^2-2x-3,\, 2x^2+x,\, -2x^2-3x-2\}.$$

We will show specializability for the case $f(x)=ax^2+(a-1)x-1$. A more extensive consideration of S_3 shows that S_∞ is not specializable for the remaining four of these polynomials. Note that for $f(x)=ax^2+(a-1)x-1$, $-f(-x-1)-1=f(x)$, so that Lemma 6 gives nothing new.

We return to the exceptional case $-1-a^2+a\,c=0$, which is solvable only for

$$(a,b,c)\in\{(1,1,2),(-1,-1,-2)\},$$

or

$$f \in \{x^2,\ -x^2-2x-2\}.$$

We will show specializability for the first of these polynomials and specializability in the second case will follow from this and Lemma 6.

For the exceptional case $1+ab-ac=0$ it is clear that $a=\pm 1$ is necessary. For $a=1$, $c=b+1$ and an examination of the third partial quotient in S_2 shows $b\in\{0,1,2\}$ is necessary. Consideration of S_4 eliminates $b=0$ and $b=2$ (using Lemma 5) and $b=1$ gives $f(x)=x^2$ (encountered above). For $a=-1$, $c=b-1$ and an examination of the third partial quotient in S_2 shows $b\in\{0,-1,-2\}$ is necessary. Lemma 5 and consideration of S_4 eliminate $b=0$ and $b=2$. The case $b=-1$ gives $f(x)=-x^2-2x-2$ (encountered above).

Lastly, for the exceptional case $-1+a^2-2\,a\,b+a\,c=0$, it is obvious that $a=\pm 1$ is again necessary, and in each case $c=2b$. Consideration of S_3 in the case $a=1$ shows that $b\in\{0,1,2\}$ is necessary. Looking at S_4 eliminates $b=0$ and $b=2$ and $b=1$ gives $f(x)=x^2$, which has been encountered above. Likewise, the case $a=-1$ necessitates $b\in\{0,-1,-2\}$. Only $b=-1$ is of interest, giving once again $f(x)=-x^2-2x-2$.

The reasoning above leads to the following theorem.

Theorem 2. *Let $f(x)\in\mathbb{Z}[x]$ be a polynomial of degree two such that $\prod_\infty(f)$ has a specializable continued fraction expansion. Then*

$$f(x)\in\{x^2,\ -x^2-2x-2,\ 2\,x^2-1,\ -2x^2-4\,x-2,\ a\,x^2+(a-1)x-1\}. \tag{4.4}$$

Proof. The necessity of (4.4) has already been shown. Also, by Lemma 6, it is enough to show sufficiency for the first, third and fifth of the polynomials in this list.

(i) If $f(x)=x^2$, then

$$\begin{aligned}\prod_{i=0}^{n}\left(1+\frac{1}{f_j}\right) &= \prod_{i=0}^{n}\left(1+\frac{1}{x^{2^j}}\right)\\ &= \frac{\sum_{j=0}^{2^n} x^j}{x^{2^n}}\\ &= \frac{x^{2^n+1}-1}{x^{2^n}(x-1)}\\ &= \left[1;x-1,\frac{x^{2^n}-1}{x-1}\right],\end{aligned}$$

which is clearly specializable for $x\neq 1$, and $S_\infty=[1;x-1]$.

(ii) If $f(x) = 2x^2 - 1$, then

$$
\begin{aligned}
S_1 &= [1; x - 1/2, -4x - 2], \\
S_2 &= [1; x - 1/2, -4x, x, -4x - 2], \\
S_3 &= [1; x - 1/2, -4x, x, -4x, x, -4x, x, -4x - 2].
\end{aligned} \tag{4.5}
$$

We will show that if $S_n = [1;\ x - 1/2,\ \vec{\omega}_n,\ -4x - 2]$, with $\vec{\omega}_n$ specializable, then

$$S_{n+1} = [1;\ x - 1/2,\ \vec{\omega}_n,\ -4x,\ x,\ \vec{\omega}_n,\ -4x - 2].$$

This can be seen to be true for $n = 1$ and $n = 2$. Let T_{n+1} denote the continued fraction which we claim is equal to S_{n+1}. By induction $\vec{\omega}_n$ is made up of the pair of terms $-4x$, x repeated a certain number of times, and if $T_{n+1} = S_{n+1}$, then it is easy to see that $\vec{\omega}_{n+1}$ will have the same form. We will also show, for $i \geq 2$, that $A_i = (1 + x)2^{i+1} \prod_{j=0}^{i-1} f_j$ and

$$
\begin{pmatrix} A_i & A_i' \\ B_i & B_i' \end{pmatrix} = \begin{pmatrix} A_i & \dfrac{f_i}{2} - \dfrac{A_i}{4} \\ 2f_i & \dfrac{f_i^2 - 1}{A_i} - \dfrac{f_i}{2} \end{pmatrix}. \tag{4.6}
$$

This is easily checked for $i = 2$ from (4.5). Suppose it is true for $i = 2, \ldots, n$.

The continued fraction T_{n+1} can be constructed as follows: take S_n, remove the final term $-4x - 2$, add the terms $-4x$ and x and then append another copy of S_n which has the first two terms (1 and $x - 1/2$) removed. Thus, by the correspondence between continued fractions and matrices which we have used several times already,

$$
\begin{aligned}
T_{n+1} &\sim \begin{pmatrix} A_n & A_n' \\ B_n & B_n' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 4x + 2 \end{pmatrix} \begin{pmatrix} -4x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad \times \begin{pmatrix} 0 & 1 \\ 1 & -x + 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A_n & A_n' \\ B_n & B_n' \end{pmatrix} \\
&= \begin{pmatrix} A_n & A_n' \\ B_n & B_n' \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} A_n & A_n' \\ B_n & B_n' \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{A_n\,(A_n + B_n + 4\,A_n')}{2} & \dfrac{A_n\,A_n' + 4\,{A_n'}^2 + A_n\,B_n'}{2} \\ \dfrac{A_n\,B_n + B_n^2 + 4\,A_n\,B_n'}{2} & \dfrac{B_n\,A_n' + B_n\,B_n' + 4\,A_n'\,B_n'}{2} \end{pmatrix} \\
&= \begin{pmatrix} 2\,A_n\,f_n & \dfrac{-1 - A_n\,f_n + 2\,{f_n}^2}{2} \\ 2\,\left(-1 + 2\,{f_n}^2\right) & \dfrac{-\left(-A_n + 4\,f_n + 2\,A_n\,{f_n}^2 - 4\,{f_n}^3\right)}{2\,A_n} \end{pmatrix} \\
&=: \begin{pmatrix} C_{n+1} & C_{n+1}' \\ D_{n+1} & D_{n+1}' \end{pmatrix}.
\end{aligned}
$$

The next-to-last equality comes from substituting for A'_n, B_n and B'_n from (4.6). Next,

$$\frac{C_{n+1}}{D_{n+1}} = \frac{2\,A_n\,f_n}{2\,\left(-1+2\,{f_n}^2\right)} = \frac{A_n}{B_n}\frac{2f_n^2}{\left(-1+2\,{f_n}^2\right)} = \prod_n \left(1+\frac{1}{f_{n+1}}\right) = \prod_{n+1},$$

so that $T_{n+1} = S_{n+1}$. Here we have also used the fact that $B_n = 2f_n$. It is also now easy to check that (4.6) now holds with $i = n+1$, so that the induction continues. Thus

$$S_\infty = [1;\, x - 1/2, \overline{-4x,\, x}\,]$$

and all that remains is to show that the expansion can be manipulated to remove the "1/2" from the first partial quotient. This follows from the identity

$$\left[x + \frac{1}{a};\, c,\, \alpha\right] = \left[x;\, a,\, -\frac{c+a}{a^2}, -a^2\alpha\right]. \tag{4.7}$$

If this identity is applied repeatedly, it follows that

$$\begin{aligned}
\prod_\infty &= [1;\, x-1/2,\, -4x,\, x,\, -4x,\, x\,-\,4x,\, x,\, -4x,\, x,\, \ldots] \\
&= [1;\, x,\, -2,\, x+1/2,\, -4x,\, x,\, -4x,\, x,\, -4x,\, x,\, -4x,\, \ldots] \\
&= [1;\, x,\, -2,\, x,\, 2,\, x-1/2,\, -4x,\, x,\, -4x,\, x,\, -4x,\, x,\, -4x,\, \ldots] \\
&\ \ \vdots \\
&= [1;\, \overline{x,\, -2,\, x,\, 2}\,],
\end{aligned}$$

which is specializable. This completes the proof for $f(x) = 2x^2 - 1$.

(iii) If $f(x) = a\,x^2 + (a-1)x - 1$, then

(4.8)

$$\begin{aligned}
S_1 &= [1; x - 1/a], \\
S_2 &= [1; x - 1/a,\, -a^3x^2 - a^3x + a], \\
S_3 &= [1; x - 1/a, -a^3x^2 - a^3x + a,\, ax^2 + (a-2)x - 1 + 1/a], \\
S_4 &= [1; x - 1/a,\, -a^3x^2 - a^3x + a,\, ax^2 + (a-2)x - 1 + 1/a, \\
&\quad - a^3\,x\,(1+x)\,\left(-1 - a\,x + a^2\,x + a^2\,x^2\right)\,\left(-1 - a - a\,x + a^2\,x + a^2\,x^2\right)].
\end{aligned}$$

The situation is somewhat similar to case (ii) in Theorem 1 (going from $\prod_n$ to $\prod_{n+1}$ adds one new term to the continued fraction expansion), but the presence of the $1/a$ term in some partial quotients is troublesome, necessitating a different approach.

Define $\alpha_1, \ldots, \alpha_4$ by

$$S_4 = [1; \alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4],$$

and for $n \geq 2$, define

(4.9)
$$\alpha_{2n+1} = \alpha_3 \prod_{i=1}^{n-1} (a\, f_{2i} - 1)(a(f_{2i}+1) - 1) = \alpha_3 \prod_{i=1}^{n-1} \frac{f_{2i+1}(f_{2i+1}+1)}{(f_{2i}+1)f_{2i}},$$

(4.10)
$$\alpha_{2n+2} = \alpha_4 \prod_{i=1}^{n-1} (a\, f_{2i+1} - 1)(a(f_{2i+1}+1) - 1) = \alpha_4 \prod_{i=1}^{n-1} \frac{f_{2i+2}(f_{2i+2}+1)}{(f_{2i+1}+1)f_{2i+1}}.$$

The second equalities follow from the definition of $f(x)$. It is clear from these definitions and (4.8) that, for $n \geq 1$, $\alpha_{2n+2}/a^3 \in \mathbb{Z}[x,a]$ and $\alpha_{2n+1} - 1/a \in \mathbb{Z}[x,a]$. We will show that

$$S_n = [1; \alpha_1, \ldots, \alpha_n], \tag{4.11}$$

for each integer $n \geq 1$. Let A_n/B_n denote the final convergent of the right side of (4.11). As part of the proof, we will show that, for $n \geq 1$,

$$\begin{aligned}
A_{2n+1} &= A_1(-1)^n \prod_{i=1}^{n} (a(f_{2i}+1) - 1) = A_1(-1)^n \prod_{i=1}^{n} \frac{f_{2i+1}+1}{f_{2i}}, \\
A_{2n+2} &= A_2(-1)^n \prod_{i=1}^{n} (a(f_{2i+1}+1) - 1) = A_2(-1)^n \prod_{i=1}^{n} \frac{f_{2i+2}+1}{f_{2i+1}}, \\
B_{2n+1} &= B_1(-1)^n \prod_{i=1}^{n} (a\, f_{2i} - 1) = B_1(-1)^n \prod_{i=1}^{n} \frac{f_{2i+1}}{f_{2i}+1}, \\
B_{2n+2} &= B_2(-1)^n \prod_{i=1}^{n} (a\, f_{2i+1} - 1) = B_2(-1)^n \prod_{i=1}^{n} \frac{f_{2i+2}}{f_{2i+1}+1}.
\end{aligned} \tag{4.12}$$

Once again the second equalities follow in each case from the form of $f(x)$. With these values, we have, for $n \geq 1$, that

$$\begin{aligned}
\frac{A_{2n+1}}{B_{2n+1}} &= \frac{A_1}{B_1} \prod_{i=1}^{n} \frac{(f_{2i+1}+1)(f_{2i}+1)}{f_{2i+1} f_{2i}} \\
&= \frac{A_1}{B_1} \prod_{i=2}^{2n+1} \left(1 + \frac{1}{f_i}\right) \\
&= \prod_{2n+1}.
\end{aligned}$$

Similarly,

$$\frac{A_{2n+2}}{B_{2n+2}} = \prod_{2n+2},$$

for $n \geq 1$. Thus to prove (4.11) it is sufficient to prove (4.12). It is not difficult to check that (4.12) holds for $n = 1$. Suppose it holds for $n =$

$1, 2, \ldots, m$.

$$
\begin{aligned}
A_{2m+3} &= \alpha_{2m+3} A_{2m+2} + A_{2m+1} \\
&= \alpha_3 \prod_{i=1}^{m} \frac{f_{2i+1}(f_{2i+1}+1)}{(f_{2i}+1)f_{2i}} \times A_2 (-1)^m \prod_{i=1}^{m} \frac{f_{2i+2}+1}{f_{2i+1}} \\
&\qquad\qquad + A_1 (-1)^m \prod_{i=1}^{m} \frac{f_{2i+1}+1}{f_{2i}} \\
&= (-1)^m \prod_{i=1}^{m} \frac{f_{2i+1}+1}{f_{2i}} \left(\alpha_3 A_2 \frac{f_{2m+2}+1}{f_2+1} + A_1 \right) \\
&= (-1)^m \prod_{i=1}^{m} \frac{f_{2i+1}+1}{f_{2i}} \left(-a\, A_1 (f_{2m+2}+1) + A_1 \right) \\
&= (-1)^{m+1} \prod_{i=1}^{m+1} \frac{f_{2i+1}+1}{f_{2i}}.
\end{aligned}
$$

The next-to-last equality follows from the fact that

$$
\frac{\alpha_3 A_2}{f_2+1} = -a\, A_1, \tag{4.13}
$$

and the last equality from the fact that $f_{2m+3}+1 = f_{2m+2}(a(f_{2m+2}+1)-1)$.

The proof that A_{2m+4} has the form stated by (4.12) is similar, except that we use the fact that

$$
\frac{\alpha_4 A_1}{f_2} = a\, A_2, \tag{4.14}
$$

The proofs that B_{2m+3} and B_{2m+4} have the forms stated by (4.12) are similar, except that we use, in turn, the facts that

$$
\begin{aligned}
\frac{\alpha_3 B_2}{f_2} &= -a\, B_1, \\
\frac{\alpha_4 B_1}{f_2+1} &= a\, B_2.
\end{aligned} \tag{4.15}
$$

This completes the proof of (4.11). What remains is to show is that S_∞ can be transformed into a specializable continued fraction. It is clear from (4.8) and the remarks following (4.9) that we can write

$$
S_\infty = \\
\left[1; x - \frac{1}{a}, -a^3(x^2+x)+a, \beta_3 + \frac{1}{a}, a^3\beta_4, \ldots, \beta_{2n+1} + \frac{1}{a}, a^3\beta_{2n+2}, \ldots \right],
$$

where each $\beta_i \in \mathbb{Z}[a,x]$. Proof of specialization now easily from a single application of (4.7), starting with the first partial quotient.

$$
\begin{aligned}
&S_\infty\\
&= \left[1; x+\frac{1}{-a}, -a^3(x^2+x)+a, \beta_3+\frac{1}{a}, a^3\beta_4, \dots, \beta_{2n+1}+\frac{1}{a}, a^3\beta_{2n+2}, \dots\right]\\
&= \left[1; x,\, (-a), -\frac{-a^3(x^2+x)+a+(-a)}{(-a)^2}, -(-a)^2\left(\beta_3+\frac{1}{a}\right), \frac{a^3\beta_4}{-(-a)^2}, \right.\\
&\qquad\qquad \left. \dots, -(-a)^2\left(\beta_{2n+1}+\frac{1}{a}\right), \frac{a^3\beta_{2n+2}}{-(-a)^2}, \dots\right]\\
&= \left[1; x,\, -a, a(x^2+x), -a^2\beta_3-a, -a\beta_4, \dots, -a^2\beta_{2n+1}-a, -a\beta_{2n+2}, \dots\right],
\end{aligned}
$$

which is specializable. This completes the proof of Theorem 2. □

5. Specialization and Transcendence

In what follows, we assume $f(x) \in \mathbb{Z}[x]$ and $M \in \mathbb{Z}$ are such that $f_j(M) \neq 0, -1$, for $j \geq 0$ and $f_i(M) \neq f_j(M)$ for $i \neq j$.

For any of the polynomials f in Theorems 1 and 2, $S_\infty(f)$ will typically have some partial quotients which are polynomials in x with negative leading coefficients. It may also happen that if $S_\infty(f)$ is specialized by letting x assume integral values, that negative or zero partial quotients may appear in the resulting continued fraction. These are easily removed, as the following equalities show (see also [21]).

$$
\begin{aligned}
[\dots, a, b, 0, c, d, \dots] &= [\dots, a, b+c, d, \dots],\\
[\dots, a, -b, c, d, e, \dots] &= [\dots, a-1, 1, b-1, -c, -d, -e, \dots]
\end{aligned}
$$

Thus, if M is an integer, repeated application of the identities above will transform $S_\infty(f(M))$ to produce the regular continued fraction expansion of the corresponding real numbers.

A natural question is whether these numbers are transcendental or not. We will make use of Roth's Theorem.

Theorem 3. (*Roth* [14]) *Let α be an algebraic number and let $\epsilon > 0$. Then the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

has only finitely many solutions with $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

We have the following theorem for the case where the degree of $f(x)$ is at least three.

Theorem 4. *Let $f(x) \in \mathbb{Z}[x]$ and $M \in \mathbb{Z}$ be such that $f_j(M) \neq 0, -1$, for $j \geq 0$ and $f_i(M) \neq f_j(M)$ for $i \neq j$.*

If either $\deg(f) > 3$ *or* $\deg(f) = 3$ *and either* $x \mid (f+1)$ *or* $(x+1) \mid f$, *then*

$$\prod_{i=0}^{\infty} \left(1 + \frac{1}{f_i(M)}\right)$$

is transcendental.

Proof. Let f and M satisfy the conditions stated in the theorem and suppose that $\deg(f) = d$ and that

$$f(x) = L\,x^d + a_1 x^{d-1} + \cdots + a_{d-1}x + a_d =: L\,x^d \left(1 + \frac{\beta(x)}{x}\right).$$

Define $\beta_i := \beta(f_i(M))$ so that $|\beta_i| \leq \sum_{i=1}^{d} |a_i|$ for all i and M. Then for $k \geq 1$,

$$\begin{aligned} f_k(M) &= L\,(f_{k-1}(M))^d \left(1 + \frac{\beta_{k-1}}{f_{k-1}(M)}\right) \\ &= L^{\frac{d^k-1}{d-1}} M^{d^k} \prod_{i=0}^{k-1} \left(1 + \frac{\beta_i}{f_i(M)}\right)^{d^{k-1-i}}. \end{aligned}$$

Note that the second equality for $f_k(M)$ also holds for $k = 0$, upon taking, as usual, the empty product to be equal to 1. Also,

$$\prod_{k=0}^{N} f_k(M) = L^{\frac{1}{d-1}\left(\frac{d^{N+1}-1}{d-1} - (N+1)\right)} M^{\frac{d^{N+1}-1}{d-1}} \prod_{i=0}^{N-1} \left(1 + \frac{\beta_i}{f_i(M)}\right)^{\frac{d^{N-i}-1}{d-1}}.$$

Then

$$\frac{\left(\prod_{k=0}^{N} f_k(M)\right)^{d-1}}{f_{N+1}(M)} = L^{-(N+1)} M^{-1} \prod_{i=0}^{N} \left(1 + \frac{\beta_i}{f_i(M)}\right)^{-1}.$$

Since $f_i(M) \neq 0$ for any i and the β_i are absolutely bounded, the product on the right converges, so that

$$\frac{1}{f_{N+1}(M)} = O\left(\frac{1}{\left(\prod_{k=0}^{N} f_k(M)\right)^{d-1}}\right). \tag{5.1}$$

On the other hand, if we set $\alpha = \prod_{\infty}(f(M))$ and $p_N/q_N = \prod_N(f(M))$ in Roth's theorem, then it is not difficult to see that

$$\left|\alpha - \frac{p_N}{q_N}\right| = O\left(\frac{1}{f_{N+1}(M)}\right).$$

Since $q_N \mid \prod_{k=0}^{N} f_k(M)$, (5.1) gives that

$$\left|\alpha - \frac{p_N}{q_N}\right| = O\left(\frac{1}{q_N^{d-1}}\right).$$

If $d \geq 4$, then

$$\left|\alpha - \frac{p_N}{q_N}\right| < \frac{1}{q_N^{2+\epsilon}}$$

has infinitely many solutions for $\epsilon = 1/2$, say, and thus $\prod_\infty(f(M))$ is transcendental. If $d = 3$ and $x \,|\, (f+1)$, then $q_N \mid f_N(M)$ and since

$$f_{N+1}(M) = L\,(f_N(M))^3\left(1 + \frac{\beta_N}{f_N(M)}\right)$$

we get that

$$\left|\alpha - \frac{p_N}{q_N}\right| = O\left(\frac{1}{q_N^3}\right), \tag{5.2}$$

so that once again $\prod_\infty(f(M))$ is transcendental. The case $d = 3$ and $(x+1) \,|\, f$ is similar, in that in this case $p_N \mid (f_N(M)+1)$. Also, q_N is within a constant factor of p_N, so that (5.2) holds and Roth's theorem once more gives transcendence. □

Corollary 2. *If $f(x)$ has any of the forms in the statement of Theorem 1 and $M \in \mathbb{Z}$ is such that $f_j(M) \neq 0, -1$, for $j \geq 0$ and $f_i(M) \neq f_j(M)$ for $i \neq j$, then $\prod_\infty(f(M))$ is transcendental.*

Proof. Each polynomial in the statement of Theorem 1 satisfies the conditions of Theorem 4. □

In the proof of Theorem 4 we were able to show the transcendence of $\prod_\infty(f(M))$ when $f(x)$ had degree three only for the special cases where $x \mid (f+1)$ or $(x+1) \mid f$. If $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree three such that $x \nmid (f+1)$ and $(x+1) \nmid f$, and M is an integer such that $f_j(M) \neq 0, -1$ for any j and $f_j(M) \neq f_k(M)$ for $j \neq k$, is the infinite product

$$\prod_{j=0}^{\infty}\left(1 + \frac{1}{f_j(M)}\right)$$

transcendental? If this is false, find a counter-example.

With this question in mind, we investigated the possibility that

$$\prod_{j=0}^{\infty}\left(1 + \frac{1}{f_j(x)}\right) = \sqrt{\frac{a\,x+b}{a\,x+c}}, \tag{5.3}$$

for a polynomial $f(x) = r\,x^3 + s\,x^2 + t\,x + u \in \mathbb{Z}[x]$ and integers a, b and c. (The coefficient of x is the same in the numerator and denominator of the rational function on the right, since the infinite product on the left tends to one as x tends to infinity.) Upon replacing x by $f(x)$, dividing the new equation into the old and squaring both sides, we get

$$\left(1 + \frac{1}{x}\right)^2 \frac{a\,f(x)+b}{a\,f(x)+c} = \frac{a\,x+b}{a\,x+c}.$$

However, comparing coefficients shows that there is no polynomial $f(x)$ with integral coefficients satisfying (5.3). Interestingly, this approach does lead to the following "near miss": if $f(x) = 4x^3 + 6x^2 - 3/2$ and M is any integer different from -1, then

$$\prod_{j=0}^{\infty} \left(1 + \frac{1}{f_j(M)}\right) = \sqrt{\frac{2M+3}{2M-1}}.$$

It is not evident to the author how to extend Theorem 4 to the remaining polynomials in $\mathbb{Z}[x]$ of degree three.

For the polynomials of degree two in Theorem 2, only $f(x) = a\,x^2 + (a-1)x - 1$ needs investigation. We have shown $\prod_{\infty}(f(M))$ converges to a rational number for $f(x) = x^2$, $M \neq 1$ (and thus a similar situation holds for $f(x) = -x^2 - 2x - 2$, by Lemma 6).

For $f(x) = 2\,x^2 - 1$, $\prod_{\infty}(f(M))$ has an infinite periodic regular continued fraction expansion (after removing negatives and zeroes) when $M \neq 0, \pm 1$, and so $\prod_{\infty}(f(M))$ converges for $M \neq 0, \pm 1$ to a quadratic irrational, namely $\operatorname{sign}(M)(M+1)/\sqrt{M^2-1}$. A similar situation holds for $f(x) = -2\,x^2 - 4\,x - 2$, again by Lemma 6.

For $f(x) = a\,x^2 + (a-1)x - 1$, it is not difficult to show from (4.9) and (4.12) that if $x \neq -1, 0$ or 1 (in the case $a = 1$) or -2 (in the case $a = -1$), then

$$\lim_{n\to\infty} \frac{B_{2n+1}}{\alpha_{2n+2}} \tag{5.4}$$

can be written as a convergent infinite product. If an irrational number α has regular expansion $[a_0; a_1, \dots]$ and its N-th approximant is p_N/q_N then

$$\left|\alpha - \frac{p_N}{q_N}\right| < \frac{1}{q_N^2 a_{N+1}}, \tag{5.5}$$

for all $N \geq 0$. If all the negatives are removed from $S_\infty(f(M))$, then α_{2n+2} will increase or decrease by at most 2 to α'_{2n+2}, say. The approximant immediately before α'_{2n+2} will still be still be A_{2N+1}/B_{2N+1}. Thus (5.4) and (5.5) will give that

$$\left|\prod_{\infty}(f(M)) - \frac{A_{2N+1}}{B_{2N+1}}\right| = O\left(\frac{1}{|B_{2N+1}|^3}\right)$$

and Roth's theorem gives that $\prod_{\infty}(f(M))$ is transcendental.

We now look at some particular examples of specialization. As Cohn showed in [6], if $l \equiv 2 \mod 4$, and $T_k(x)$ denotes the k-th Chebyshev polynomial then

$$\prod_{j=0}^{\infty} \left(1 + \frac{1}{T_{l^j}(x)}\right)$$

has a specializable continued fraction expansion with predictable partial quotients. This follows from Theorem 1 (iv), using the facts that $T_1(x) =$

x, that if $l \equiv 2 \mod 4$ then $T_l(x) \equiv 2x^2 - 1 \mod x(x^2-1)$ and that $T_a(T_b(x)) = T_{ab}(x)$, for all positive integers a and b. For example, setting $l = 6$ and $x = 3$, we get after removing negatives, that

$$\prod_{j=0}^{\infty}\left(1+\frac{1}{T_{6^j}(3)}\right) = \\ [1;2,1,1632,1,2,1,3542435884041835200,1,2,1,1632,1,2,1, \\ 2602953921777123453854421658848856619640265580447716525393363412220776182840684687324960468372004114475959136 00, \\ 1,2,1,1632,1,2,1,3542435884041835200,1,2,1,1632,1,2,1,\dots].$$

In part (vi) of Theorem1, setting $g(x) = (x^{2k-2}-1)/(x^2-1)$ gives $f(x) = x^{2k}$, for $k \geq 2$, so that

$$\prod_{j=0}^{\infty}\left(1+\frac{1}{x^{(2k)^j}}\right)$$

has a specializable continued fraction expansion with predictable partial quotients. This result can also be found in [12], where the formulae for the partial quotients that we have are also given. For example, if $k = 2$ and $x \geq 2$ is a positive integer, then

$$\prod_{j=0}^{\infty}\left(1+\frac{1}{x^{4^j}}\right) = \big[1; x-1, 1, x(x-1), x(x+1), \\ x^3(x-1)(x^2+1), x^5(x+1)(x^4+1), \\ x^{11}(x-1)(x^2+1)(x^8+1), x^{21}(x+1)(x^4+1)(x^{16}+1), \dots, \\ x^{(2\times 4^i+1)/3}(x-1)\prod_{j=0}^{i-1}(x^{2\times 4^j}+1), x^{(4^{i+1}-1)/3}(x+1)\prod_{j=0}^{i}(x^{4^j}+1), \dots\big].$$

6. Concluding Remarks

Ideally, one would like to have a complete list of all classes of polynomials $f(x)$ for which $\prod_{n=0}^{\infty}(1+1/f_n)$ has a specializable continued fraction expansion. We hesitate to conjecture that our Theorems 1 and 2 give such a complete list, since there may be other classes of polynomials for which S_∞ displays more complicated forms of duplicating symmetry. One reason for suspecting this is that Cohn [6] found some quite complicated duplicating behavior for several classes of polynomials. One example he gave was the class of polynomials of the form

$$f(x) = x^3 - x^2 - x + 1 + x^2(x-1)^2 g(x),$$

with $g(x) \in \mathbb{Z}[x]$. If $S_n = \sum_{j=0}^{n} 1/f_j = [0; \vec{s}_n]$, then, for $n \geq 3$,

$$S_n = [0; \vec{s}_{n-1}, X_n, -\vec{s}_{n-2}, 0, \vec{s}_{n-4}, Y_{n-2}, 0, Z_n, -\overleftarrow{s}_{n-4}, Y_n, \overleftarrow{s}_{n-2}], \tag{6.1}$$

where the X_i, Y_i and Z_i are polynomials in $\mathbb{Z}[x]$. It is not unreasonable to suspect similar such complicated behavior might also exist in the infinite product case.

We hope the results in this paper will stimulate further work on this problem.

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