Detecting Information Loss in Shape Theory

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The Notion of Topological Space

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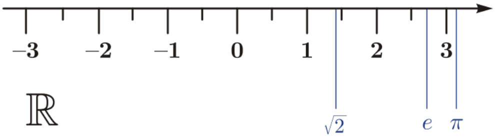
- Much of Topology deals with generalizing properties and subets of the real number line which we denote by $\mathbb{R}$.
- Topology is also concerned with studying **topological spaces** in their own right and there properties: Compactness, Connectedness/Path Connectedness, etc.

The Notion of Topological Space



The simplest example of a topological space is the real number line we are familar with from elementary algebra and calculus.

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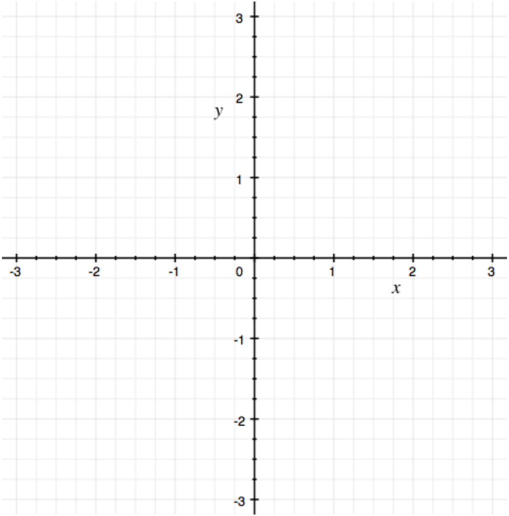

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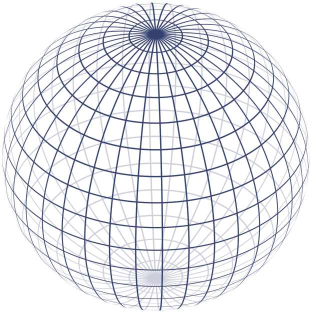
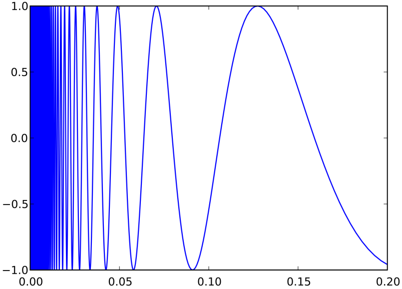
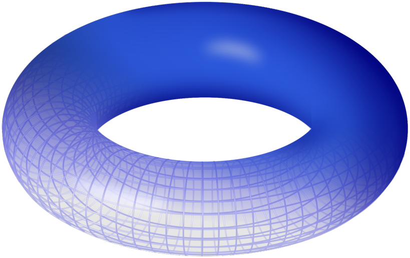






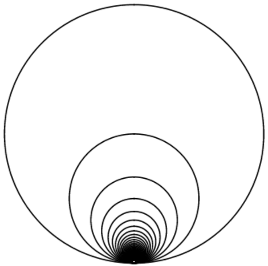


Figure: The Earring Space $\mathbb{E}$

Some spaces can be more "wild". The prototypical example of such a space is the "Earring Space" which can be realized as a union of an infinte wedge of shrinking circles that share a common point.

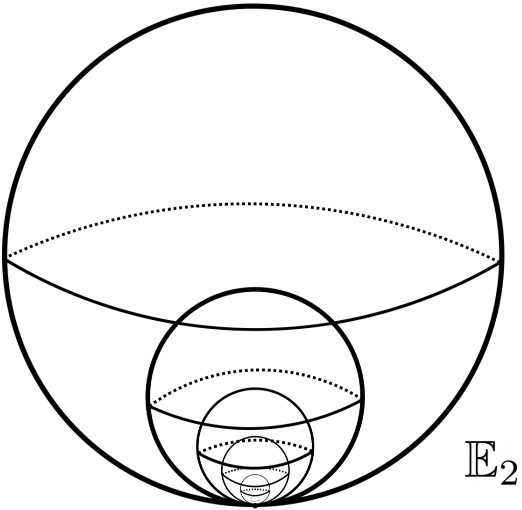


Figure: The 2D-Earring or the Barratt-Milnor Sphere (Point Wildness)

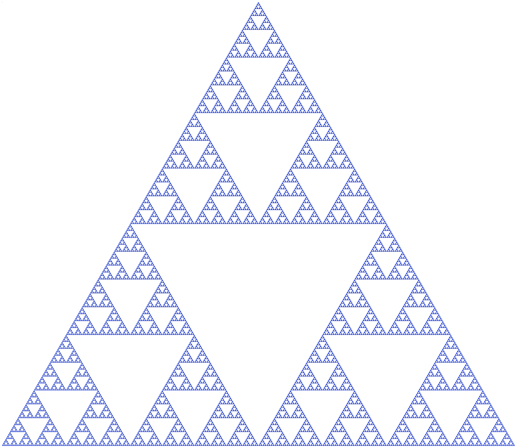


Figure: The Serpenski Gasket (Global Wildness)

Algebraic Topology

What is Algebraic Topology?: Algebraic Topology uses tools from abstract algebra to study topological spaces. More formally, the basic objective is to find algebraic invarients that calssify topological spaces up to homeomorphism (but mostly up to homotopy)

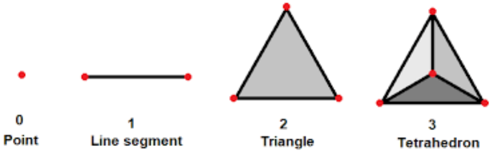
$\pi_i(S^n)$

$n \downarrow$ / $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Figure: Homotopy groups of spheres

The most important of these algebraic invarients is what is known as the **Homotopy Groups** which are denoted by $\pi_n(X, x_0)$. Here X is a topological space and x_0 is a particular point of that space. Roughly speaking, the homotopy groups capture or detect data about the shape and number of "holes" in a given space.

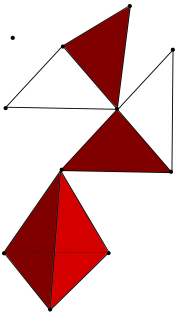
Simplicial Complexes



A **Simplex** is a genrealization of a triangle to higher dimensions. We can define the **Standard** ***n*-simplex** to be the subset of $\mathbb{R}^{n+1}$ defined by:

$$\Delta^n = \{(s_1, \ldots, s_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} s_i = 1, s_i \geq 0\}$$

Simplicial Complexes



We consider a **Simplicial Complex** $\mathcal{A}$ to be a set of simplicies that statisy the following stipulations:

1. Every face of a simplex of $\mathcal{A}$ is also in $\mathcal{A}$
2. The intersection of two simplices $\sigma, \sigma' \in \mathcal{A}$ is a face of both σ and σ'.

The Nerve of an Open Cover

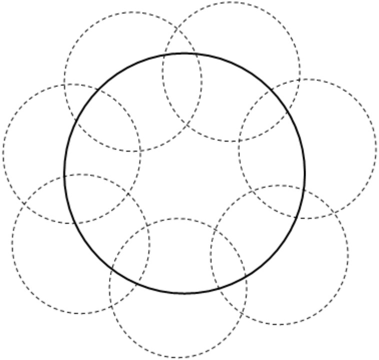


Figure: A cover of the space S^1

What's of prime importance here is that under the stipulation that X is compact, we can cover the space by a finite number of open sets; this allows us to conceptualize the **nerve of an open cover**; denoted by $N(\mathcal{U})$ where $\mathcal{U}$ is an open cover. Let us consider a general example

The Nerve of an Open Cover

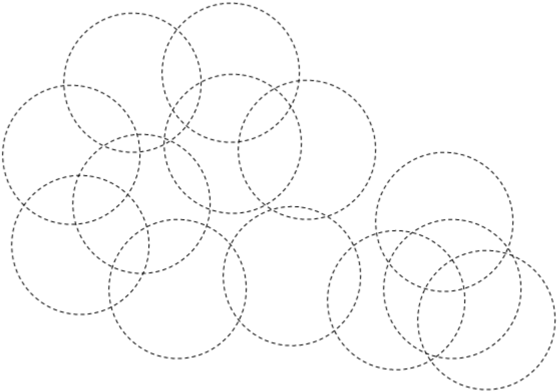


Figure: An open cover of a space X

The Nerve of an Open Cover

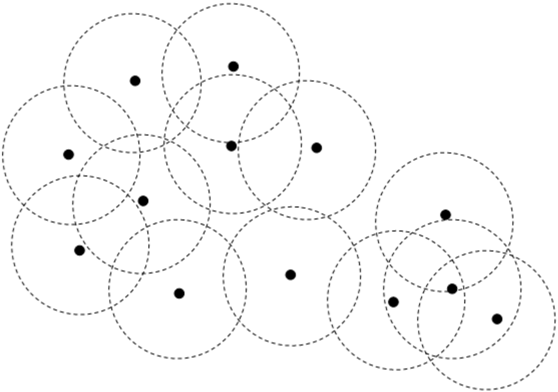


Figure: The 0-Skeleton

A vertex is placed for each U of the cover $\mathcal{U}$.

The Nerve of an Open Cover

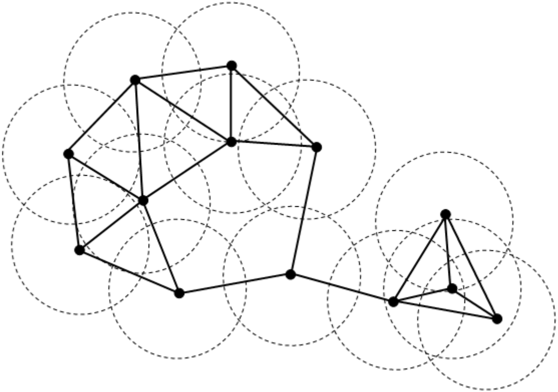
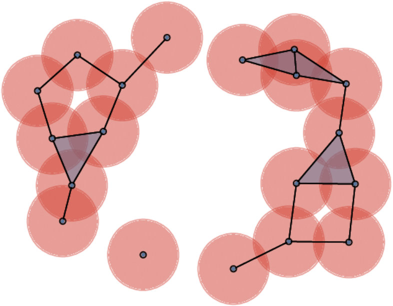


Figure: The 1-Skeleton

When $U_1 \cap U_2 \neq \emptyset$ we will place an edge (or rather, a 1-simplex) between their vertices (or each neighborhood).

The Nerve of an Open Cover

To form the 2-Skeleton we consider $U_1 \cap U_2 \cap U_3 \neq \emptyset$ and so there will be three edges joining each pair of the vertices U_1, U_2, U_3. We place a triangle (or 2-simplex) so that the edges of the triangle will match up with these three edges.



In general, we can continue forming the n-skeleton for $U_1, \ldots, U_{n+1}$ where $\bigcap_{i=1}^{n+1} U_i \neq \emptyset$ attaching an n-simplex to fill in the boundary. This is the nerve $N(\mathcal{U})$.

The Nerve of an Open Cover

We often want to consider maps $P_{\mathcal{U}} : X \to N(\mathcal{U})$ (from a topological space X to the nerve of its cover $\mathcal{U}$). In these mappings we often lose certain "geometric" information about the space in consideration. Sometimes the nerve is not "good enough" and we need to form a **refinement** of such a cover. In this process of refinements we form what is called the ***n*-th Shape Homomorphism** written:

$$\Psi^n_X : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$$

Here

$$\check{\pi}_n(X, x_0) = \varprojlim(\pi_n(|N(\mathcal{U})|, U_0), p_{\mathcal{U}\mathcal{V}\#}, \Lambda)$$

is the n-th Shape Group (or nth-Čech homotopy group).

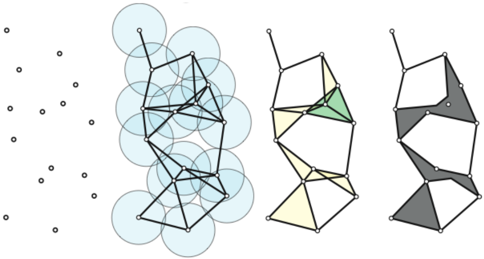
Of particular importance is the shape morphism is the **Shape Kernel** $\ker \Psi_X^n$. The shape kernel gives us a measurement of “data retainment” when passing from the n-th homotopy group to the n-th shape group. Where are particularly interested in when the shape kernel is trivial (i.e $\ker \Psi_X^n = 0$) for this tells us that X is π_n-**shape injective**. Further from this, one can realize elements of the n-th homotpy group as sequences of inverse limits of fundamental groups of polyhedra.

Detecing Information Loss In Shape Theory

Figure: Karol Borsuk - Creator of Modern Shape Theroy

The Nerve of an open cover is a concept often used in a field of math known as **Shape Theory**. In short, the goal of Shape Theory is to approximate topological spaces by nerves of open covers.In particular, we can apply the notion of the nerve of an open cover to analyze data.

Detecing Information Loss In Shape Theory



Topological Data Analysis uses the notion of nerves of open covers for analyzing discrete data sets. Previous results by Dr. Brazas and Dr. Fabel have been applied here to find the “shape” of data utilzing the methods outlined here..

n-Spanier Groups and n-Thick Spanier Groups

Figure: Dr.Edwin Spanier

The crux of our research deals with the notion of what is called a **Spanier group**. The Spanier group is a subgroup (subset that is also a group in its own right) of $\pi_1(X, x_0)$ which we denote by $\pi^{\mathrm{Sp}}(\mathcal{U}, x_0)$ which is generated by elements $[p][\gamma][p^{-}]$ where $[p]$ is a path class and $\gamma : [0, 1] \to X$ is a loop which is based at $p(1)$ for some $U \in \mathcal{U}$.

n-Spanier Groups and n-Thick Spanier Groups

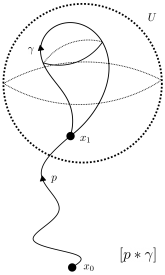


Figure: A generator of $\pi_2^{\mathrm{Sp}}(X, x_0)$

It's possible to extend the notion of the Spanier group to higher Spanier groups;for these we write $\pi_n^{\mathrm{Sp}}(X, x_0)$ and they are subgroups of $\pi_n(X, x_0)$; the n-th homotopy groups. We can define the n-Spanier group (with respect to an open cover $\mathcal{U}$) mathematically as:

$$\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) = \langle [p * \gamma] \mid p \in P(X, x_0), \mathrm{Im}(\gamma) \subseteq U, U \in \mathcal{U} \rangle$$

n-Spanier Groups and n-Thick Spanier Groups

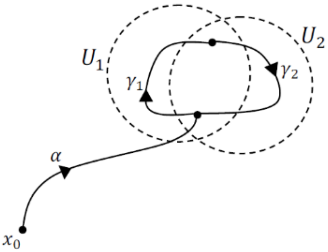


Figure: A generator of the Thick Spanier Group $\Pi_1(\mathcal{U}, x_0)$

There is also a notion of the **Thick Spanier Group**; this was developed in a paper by Dr. Jeremey Brazas, and Dr. Paul Fabel. In this paper, the first Thick Spanier group is defined as:

$$\Pi_1^{\mathrm{Sp}}(\mathcal{U}, x_0) = \langle p{\cdot}\gamma{\cdot}p^{-1} \mid p \in P(X, x_0), \gamma \in \Omega(U_1 \cup U_2, p(1)), U_1, U_2 \in \mathcal{U} \rangle$$

n-Spanier Groups, n-Thick Spanier Groups and the Shape Kernel

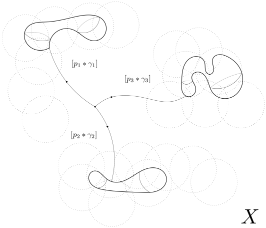


Figure: A generic element of the 2-thick Spanier group

In our work, we have sought to extend the definition of the Thick Spanier group to higher dimensions; in other words we have aimed to define $\Pi_n^{\mathrm{Sp}}(X, x_0)$. We have determined the higher thick Spanier groups can be defined as follows

$$\Pi_n(\mathcal{U}, x_0) = \langle [p*\gamma] \mid p \in P(X, x_0), \gamma \in \Omega^n(X, p(1)); I \neq \emptyset, \mathrm{Im}(\gamma) \subseteq N \rangle$$

where $I = \bigcap_{i=1}^{m} U_i, N = \bigcup_{i=1}^{m} U_i$.

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Further results and conjectures

In the future we hope to utilize these results to extend the work of Dr. Brazas and Dr. Fabel on the existence of the short exact sequence:

$$1 \to \Pi_1^{\mathrm{Sp}}(\mathcal{U}, x_0) \to \pi_1(X, x_0) \to \pi_1(|N(\mathcal{U})|, U_0) \to 1$$

which provides certain key information about the fundamental group of the nerve of a cover; we wish to extend this to:

$$0 \to \Pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) \to \pi_n(X, x_0) \to \pi_n(|N(\mathcal{U})|, U_0) \to 0$$

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Thank You!