HOW TO WIN IN CIRCULAR NIM

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INTRODUCTION TO COMBINATORIAL GAMES

WHAT IS A COMBINATORIAL GAME?

- Two-player game
- Both players have complete knowledge - no randomness or hidden information
- Typically, the last player to move wins
- If players have the same moves, then the game is called **impartial**, otherwise it is called **partisan**.

Main Question: Who wins from a given position, assuming both players play optimally?

THE GAME OF NIM

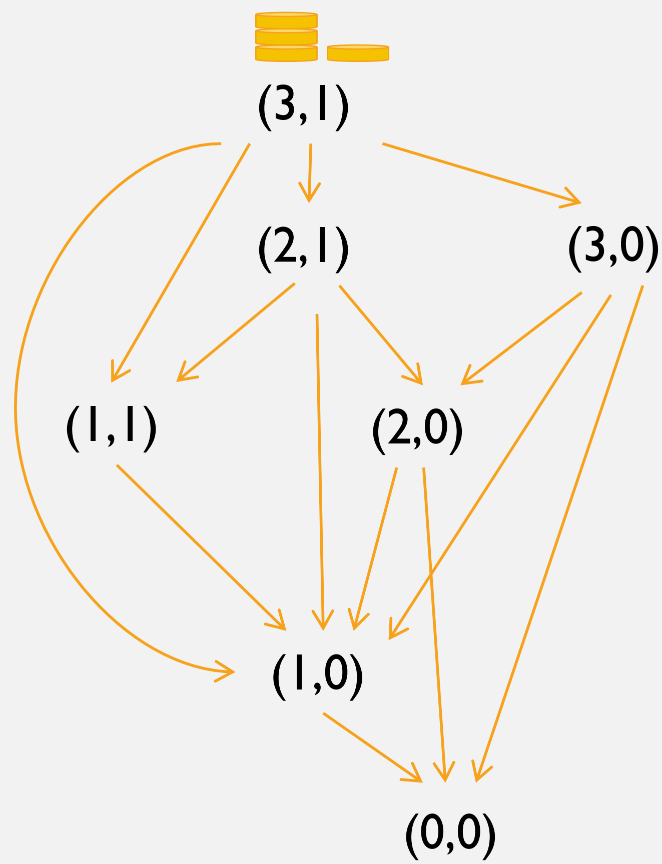
- Consists of stacks of tokens

- To move, pick one of the stacks and remove at least one token from that stack
- Last person to make a move wins

NIM is an impartial game and was completely analyzed by C. Bouton in 1901

HOW TO ANALYZE IMPARTIAL GAMES?

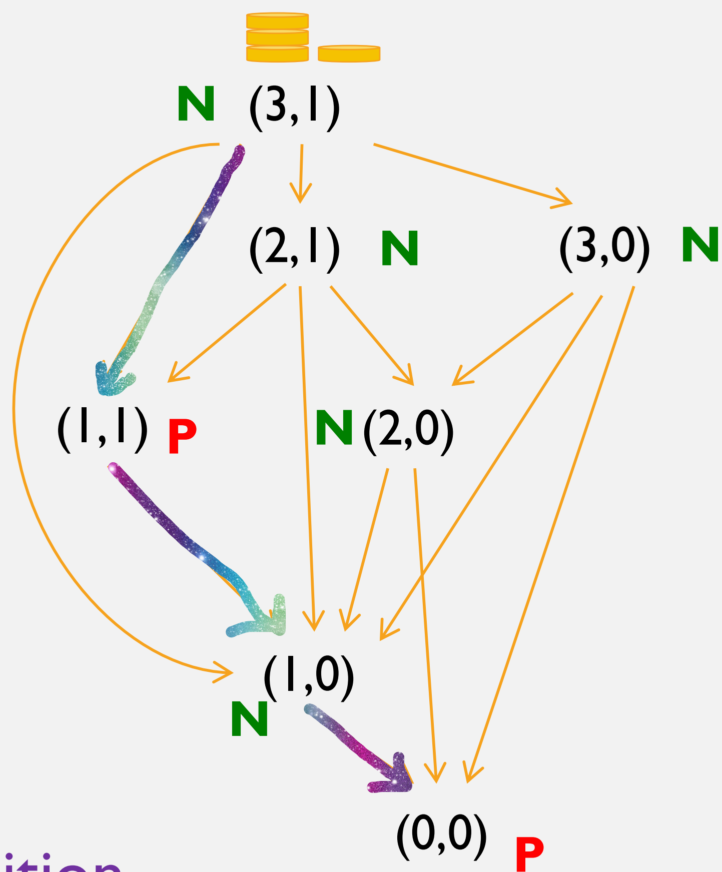
- Map out all the possible moves and counter moves
- Decide which ones are the good positions
- Map out a winning strategy



BACKWARDS LABELING OF GAME GRAPH

- Terminal positions are **P**-positions (losing)
- Any position that allows a move to a **P**-position is an **N**-position (wiNning)
- Any position for which **all** options are **N**-positions is a **P**-position.

Note: For impartial games there are only two outcome classes – N and P-positions.



Winning Strategy: Move to a P-position

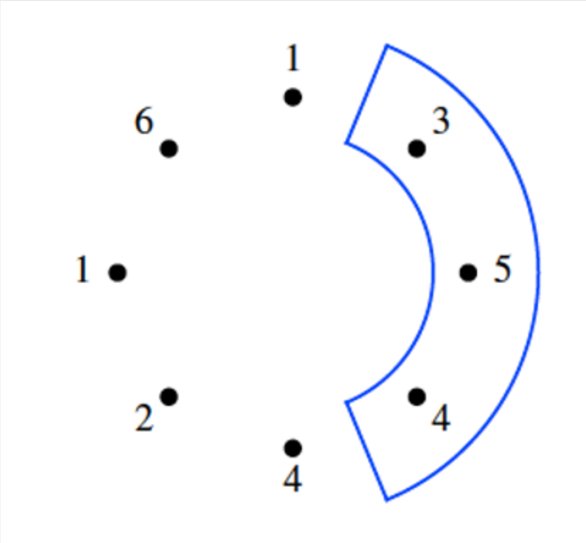
FINDING THE PATTERN OF THE P-POSITIONS

- Write a computer program to implement the labeling procedure
- Produce a list of losing positions
- Analyze them to find a **PATTERN**
- Prove that the pattern is correct by showing that:
  - there is no move from a P-position to another P-position
  - from any N-position, there is at least one move to a P-position

CIRCULAR NIM

DEFINITION OF THE GAME CN(n, k)

- n stacks of tokens arranged in a circle
- To move, select k consecutive stacks and remove at least one token from at least one of the stacks.
- Last player to move wins
- Single terminal position – (0,0,…,0)



$k = 1$ corresponds to NIM

KNOWN RESULTS

$\text{CN}(n, 1)$

$\text{CN}(n, n)$

Trivial - First player takes all tokens

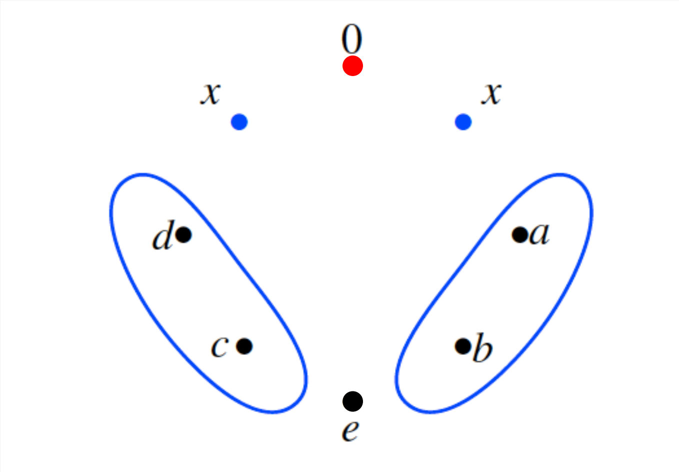
$\text{CN}(n, n-1)$

P-positions: all stack heights equal

CN(1,1)								
CN(2,1)	CN(2,2)							
CN(3,1)	CN(3,2)	CN(3,3)						
CN(4,1)	CN(4,2)	CN(4,3)	CN(4,4)					
CN(5,1)	CN(5,2)	CN(5,3)	CN(5,4)	CN(5,5)				
CN(6,1)	CN(6,2)	CN(6,3)	CN(6,4)	CN(6,5)	CN(6,6)			
CN(7,1)	?	CN(7,3)	CN(7,4)	CN(7,5)	CN(7,6)	CN(7,7)		
CN(8,1)	?	CN(8,3)	CN(8,4)	?	CN(8,6)	CN(8,7)	CN(8,8)	
CN(9,1)	?	?	?	?	?	?	CN(9,8)	CN(9,9)
CN(10,1)	?	?	?	?	?	?	?	CN(10,9)

NIM

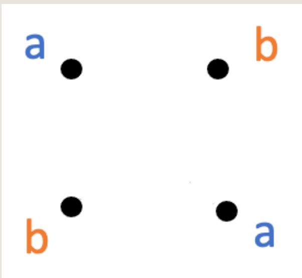
VISUAL CODING OF P-POSITIONS

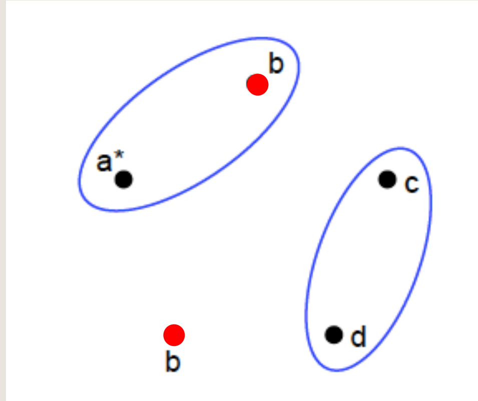


- The minimum stack(s) will be rendered with red dot
- For ovals with the same color, the sums of the respective stack heights must be equal
- If there is a stack with a blue dot and an oval in the same color, then the "blue" stack height must equal the sum of the stack heights in the oval.

RESULTS FOR INDIVIDUAL GAMES

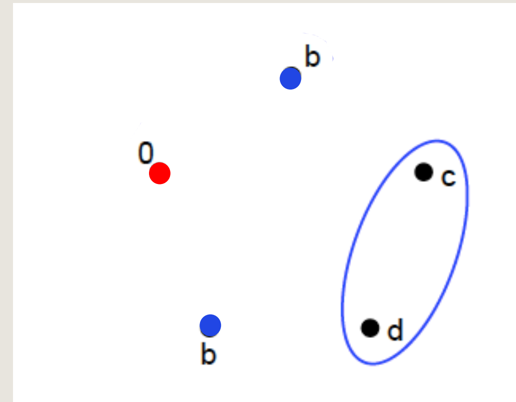
CN(4,2)



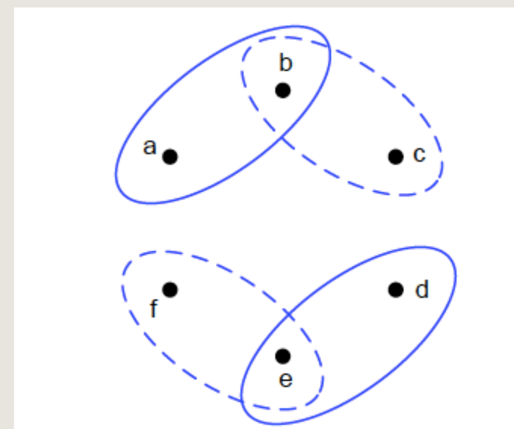


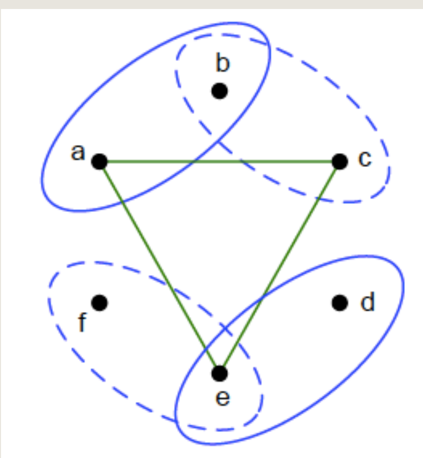
CN(5,2); a* is max

CN(5,3); b is max



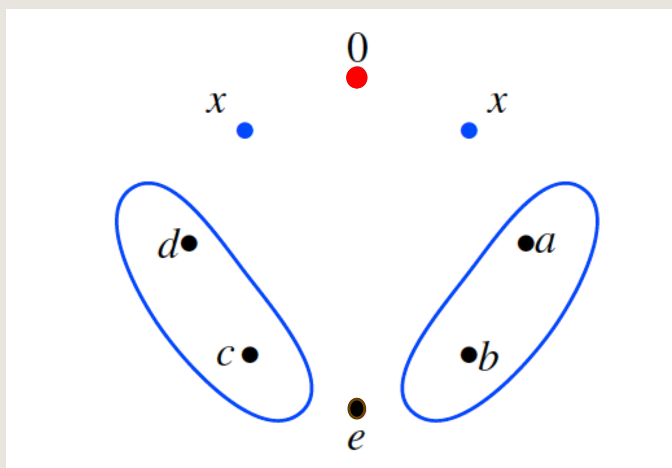
CN(6,3)





CN(6,4)

$a \oplus c \oplus e = 0$



CN(8,6);

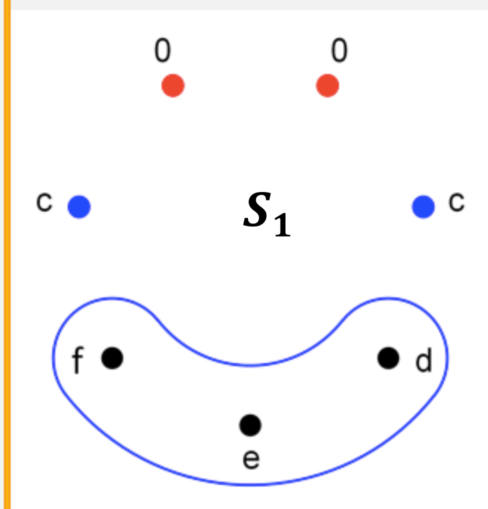
e = min { x, a + d }

P-POSITIONS OF CN(7,4)

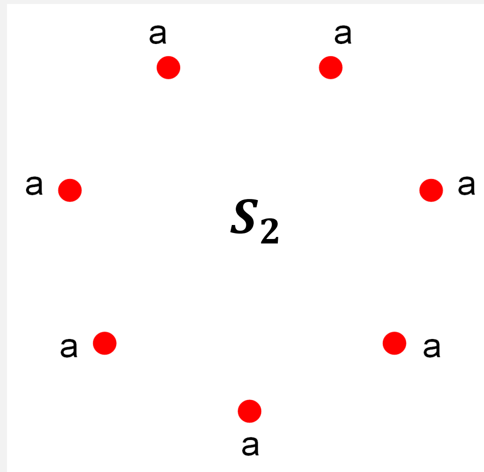
P-POSITIONS OF CN(7,4)

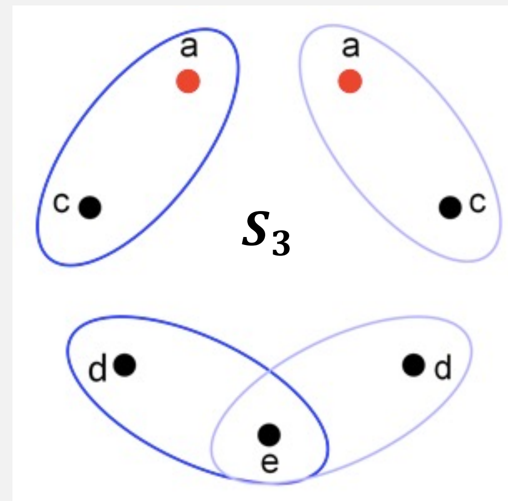
Theorem: The ***P***-positions of CN(7,4) are given by $P = S_1 \cup S_2 \cup S_3 \cup S_4$ with

Note: Terminal position (all zeros) is in S_2

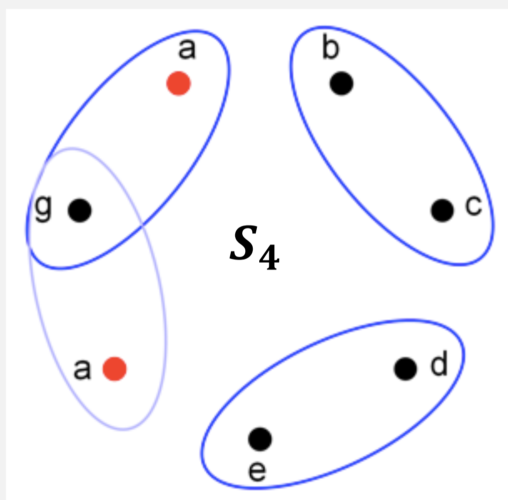


$$c > 0$$
$$d + e + f = c$$





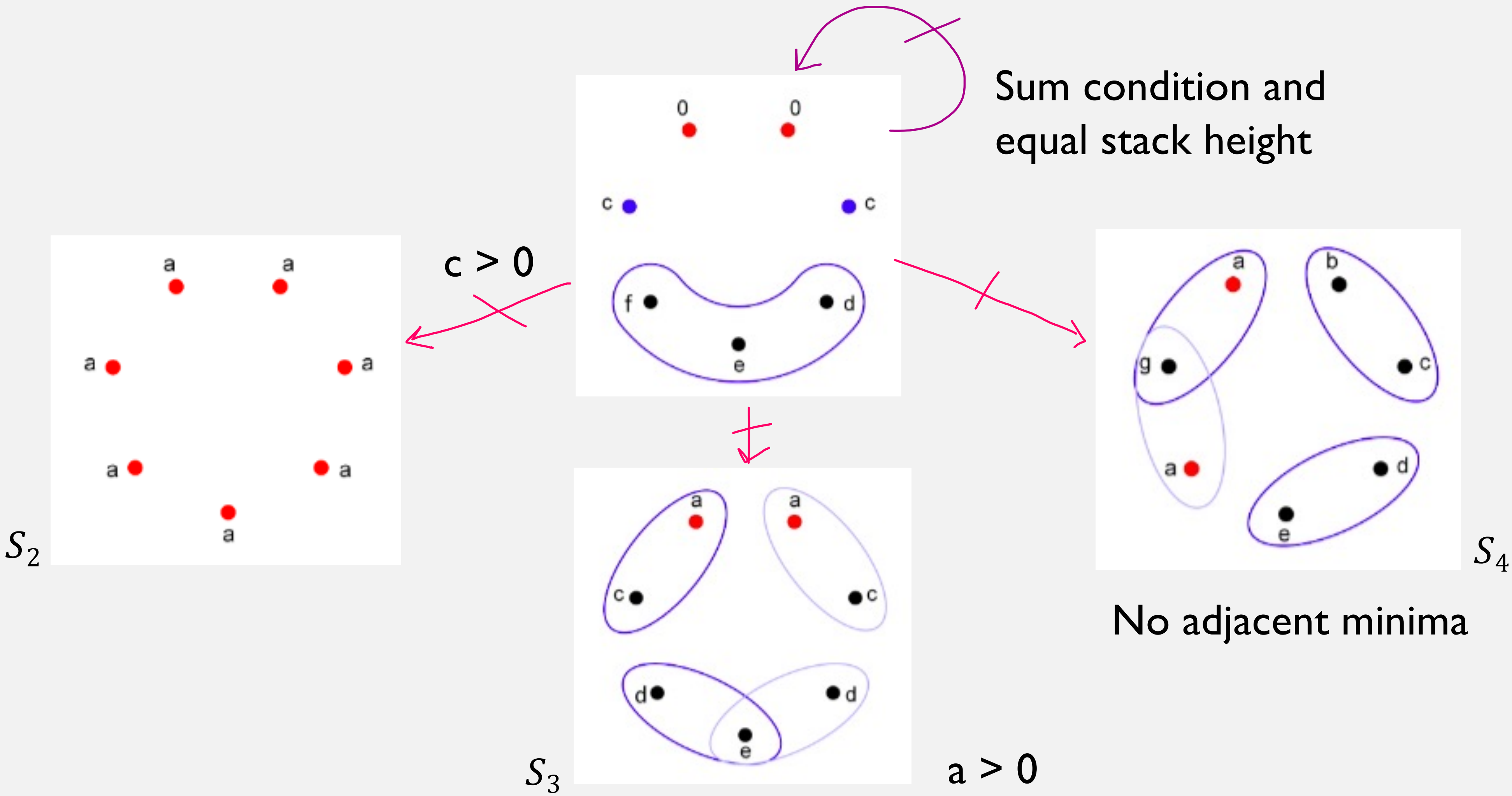
$$a + c = d + e, 0 < a < e$$



$$b + c = d + e = g + a,$$
$$a < \min\{b, e\},$$
$$a < \max\{c, d\}$$

PROOF OUTLINE CN(7,4)

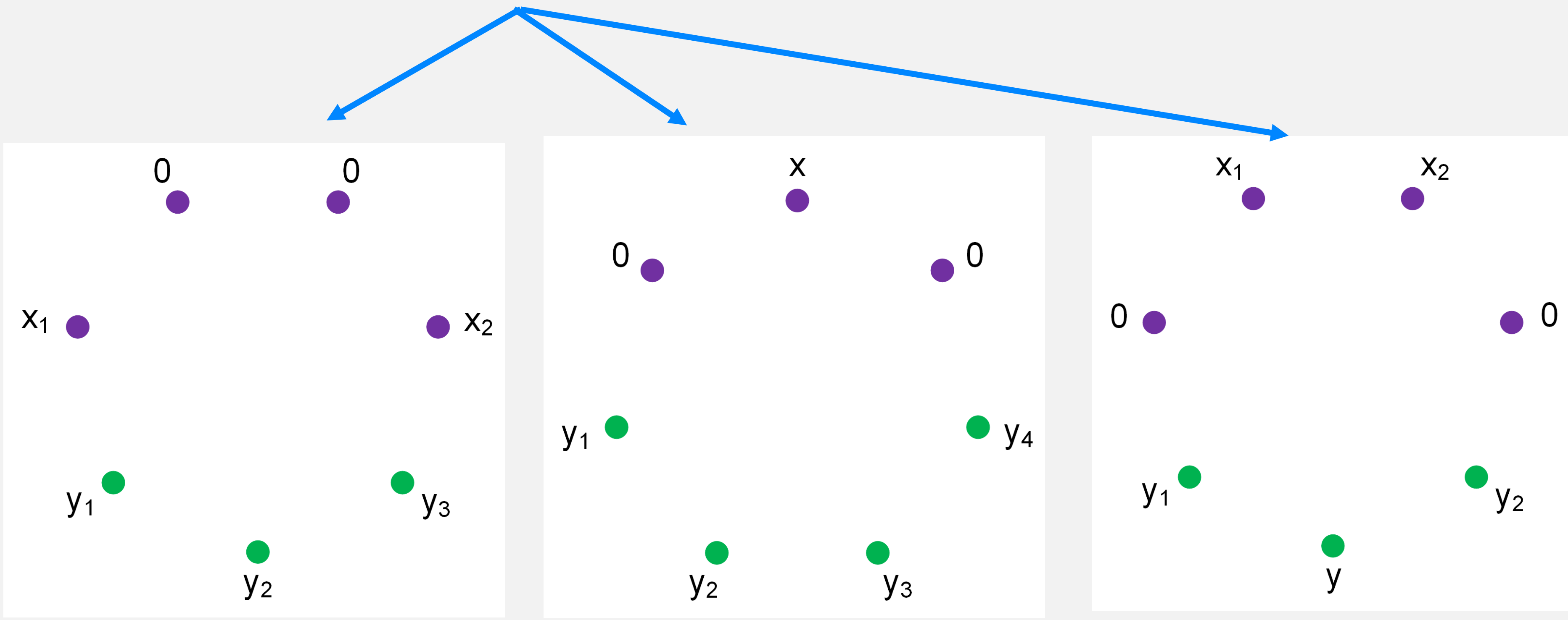
To show: Cannot move from a P-position to another $P-$position. We illustrate for S_1.



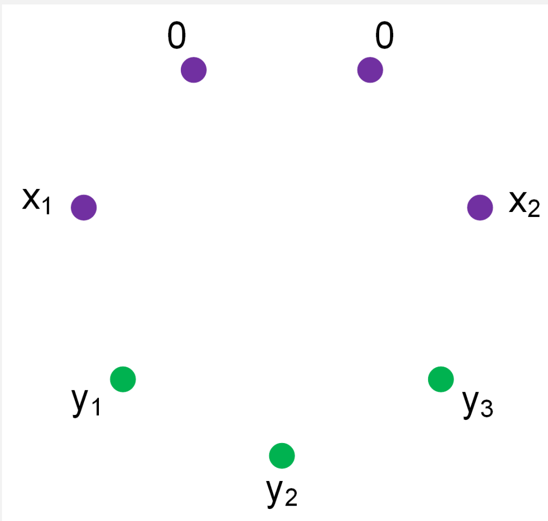
PROOF OUTLINE

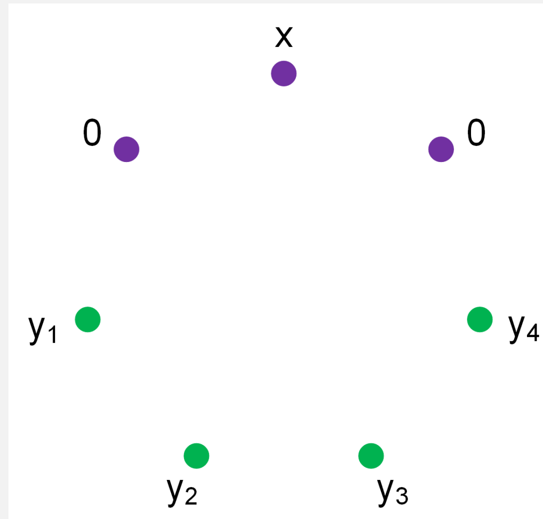
To show: From every N-position there is a move to a P-position

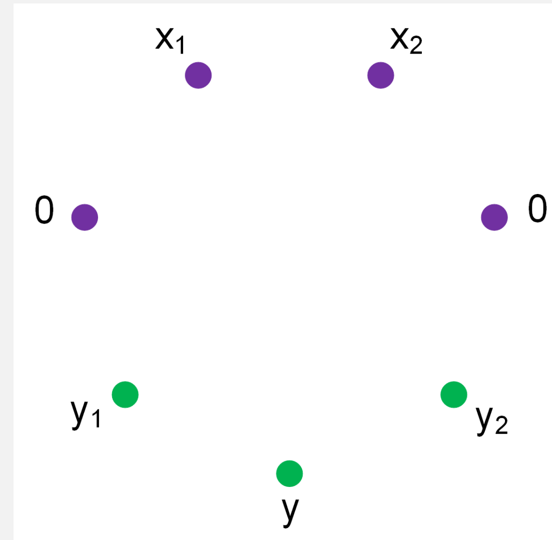
Cases: **At least two zeros**, a Unique zero, or No zero

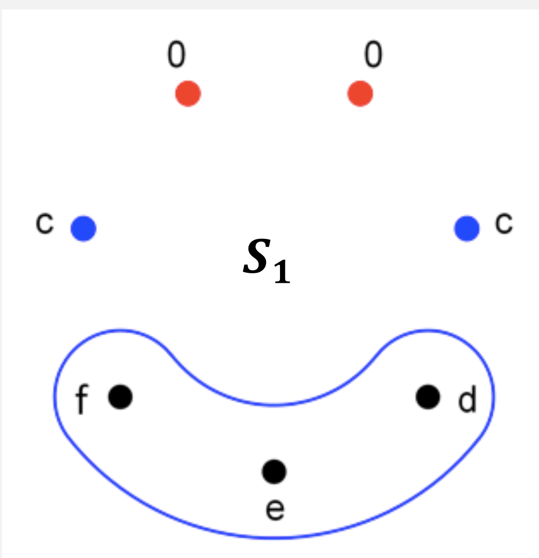


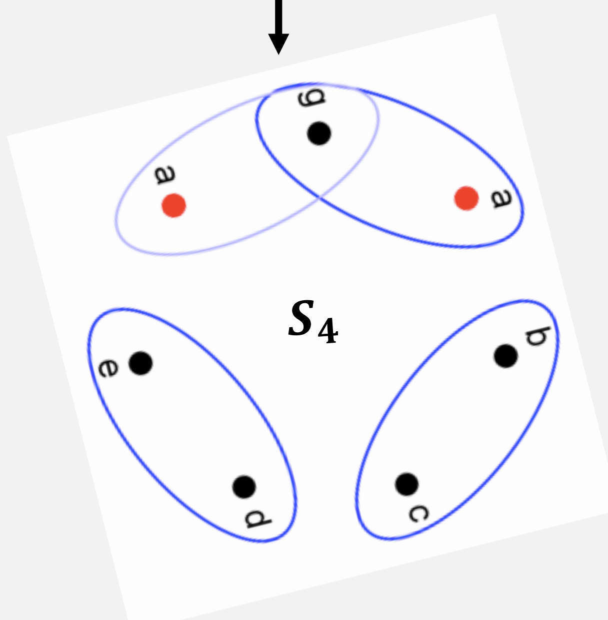
PROOF OUTLINE









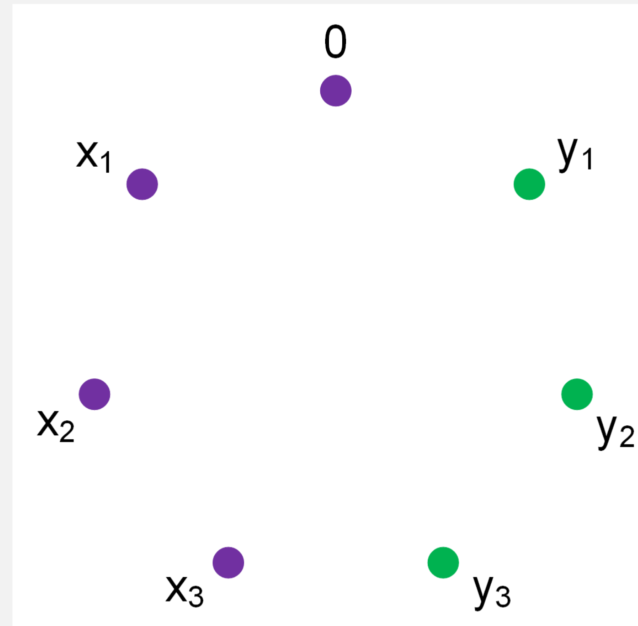


Argument more involved

PROOF OUTLINE

To show: From every N-position there is a move to a P-position

Cases: At least two zeros, a **Unique zero**, or No zero



$x_1 + y_1 \leq \min\{x_2, y_2\} = y_2$			(a)	$\boldsymbol{p}' \in S_1$
$x_1 + y_1 > y_2$	$y_2 \geq y_1$		(b)	$\boldsymbol{p}' \in S_1$
	$y_2 < y_1$	$x_2 \geq y_1$	(c)	$\boldsymbol{p}' \in S_1$
		$x_2 < y_1$	(d)	$\boldsymbol{p}' \in S_1 \cup S_4$

GAMES CN$(2l+1, l+1)$

SOME NOTES ON THE P-POSITIONS OF LARGER GAMES

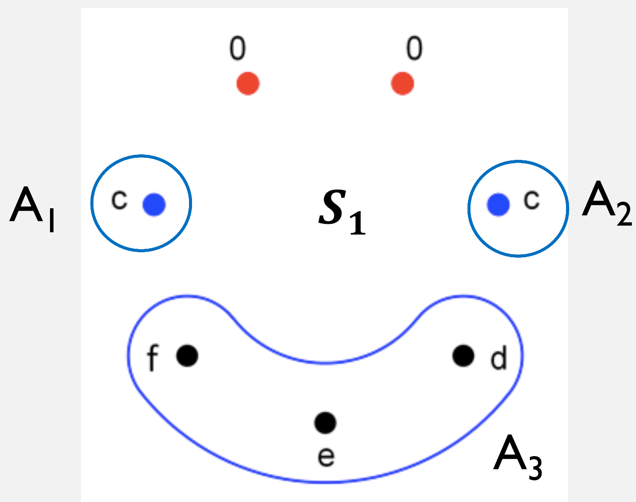
- CN(7,4) first instance with a multiple pattern description
- When we have multiple patterns, the **description** is not unique
- Multiple patterns are not a fluke – larger games contain smaller games as sub-structures

- "Reduction" can occur in different ways
- In CN(7,4), we used CN(3,2)-equivalence

CN(3,2)-EQUIVALENCE

- P-positions of CN(3,2) have equal stack heights
- CN(3,2) winning move: Leave the smallest stack untouched and reduce the two others to that height.

CN(7,4)



$$c > 0$$
$$d + e + f = c$$

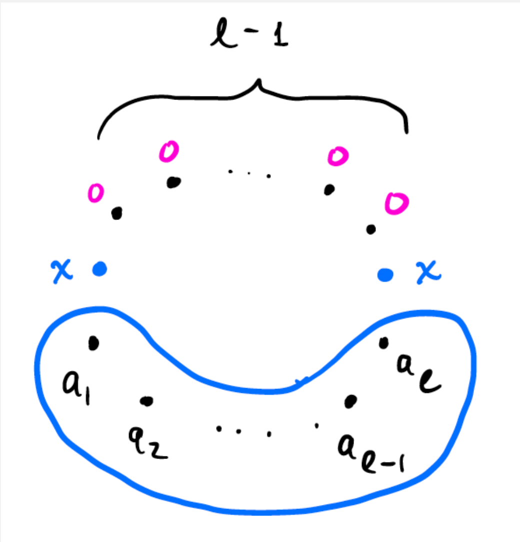
CN(3,2) equivalence generalizes to CN(m, l) equivalence:

- m disjoint sets and one or more zeros
- Play on any l sets uses at most k stacks
- Play on any $l + 1$ sets uses more than k stacks

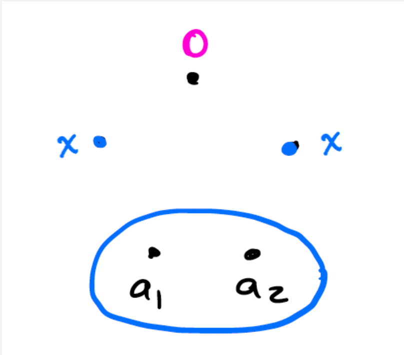
GENERAL STRUCTURE OF P-POSITIONS

Lemma: The set of P-positions of game CN$(2l+1, l+1)$ contains S_1, where

$$S_1 = \{p = (x, \underbrace{0, \dots, 0}_{l-1}, x, a_1, \dots, a_l) \mid \sum_{i=1}^{l} a_i = x\}$$



$l = 1$
CN(3,2)



$l = 2$
CN(5,3)

OPEN QUESTIONS

- Is the generalization of S_2 part of the P-positions for the family CN$(2l + 1, l + 1)$?

Answer: NO - $(2, 2, 2, 2, 2, 2, 2, 2, 2)$ is an N-position for CN(9,5).

- What are the generalization of S_3 and S_4?
- Are they part of the P-positions for the family CN$(2l + 1, l + 1)$?

GAME CN$(8,4)$

GATEWAY TO CN$(2l, l)$

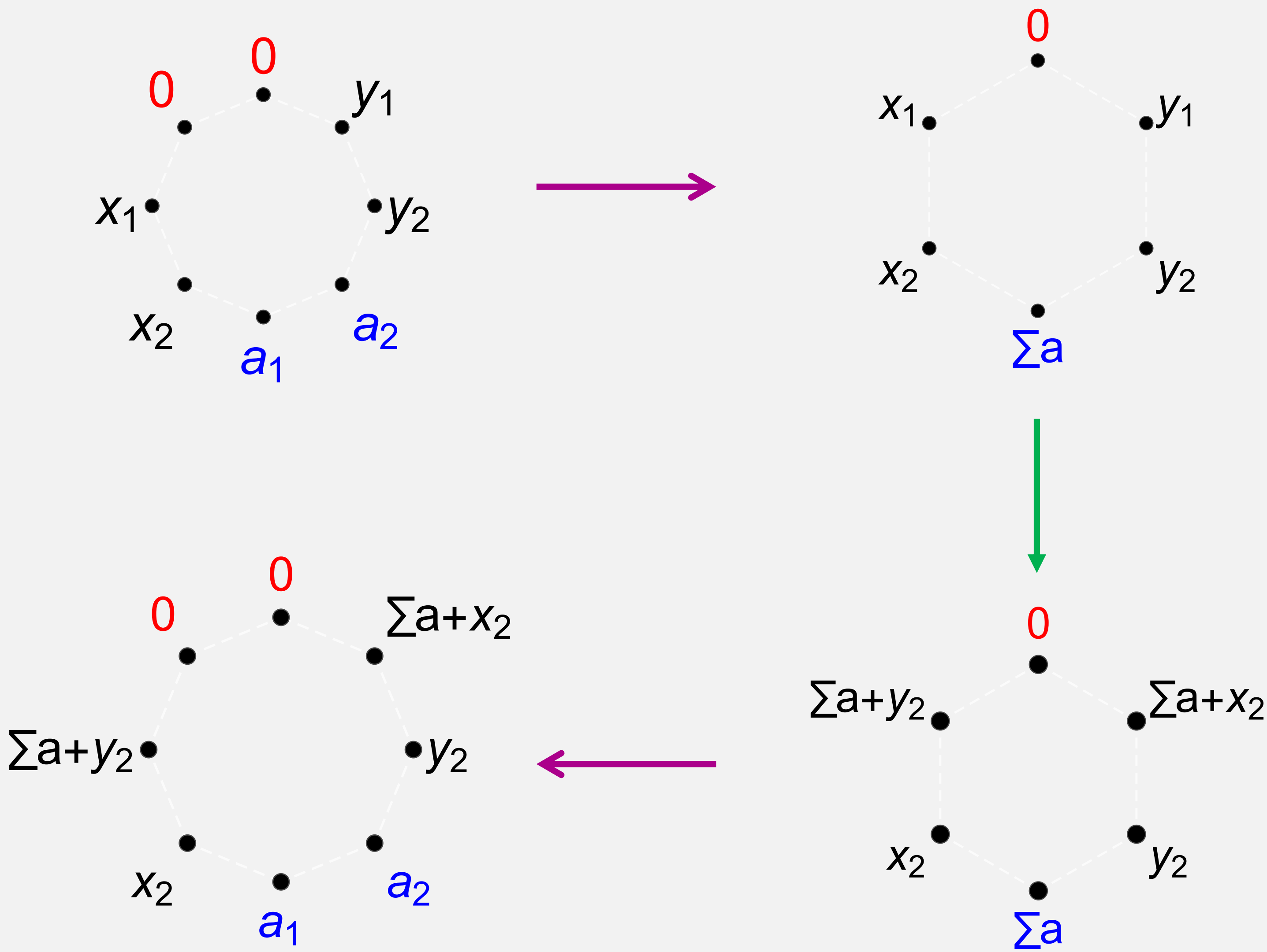
APPROACH TO FINDING P-POSITIONS

Partition (non-terminal) P-positions into those with:

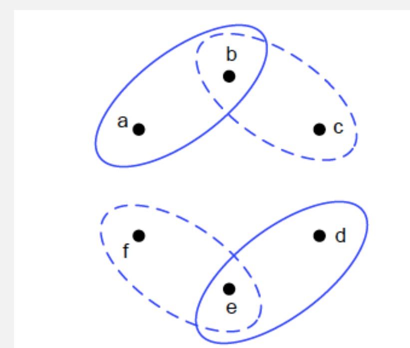
- consecutive zeros (at most $n - k - 1$)
- at least two isolated zeros
- exactly one zero
- no zeros

Lemma: The set of P-positions of the game CN$(2n, n)$ with consecutive zeros equals the set of P-positions of the game CN$(2(n-1), n-1)$ that have a zero.

P-POSITIONS OF CN(8,4) WITH AT LEAST 2 CONSECUTIVE ZEROS



P-positions in CN(6,3) with zero



REMAINING CASES FOR CN(8,4)

- We have **partial** patterns for the case of at least two isolated zeros

- Case of exactly one zero ….
- Case of no zeros….

THANK YOU!

Any questions?

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REFERENCES

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- Dufour, Matthieu, Heubach, Silvia and Vo, Anh. Circular Nim games CN(7, 4), *Integers*, **21B** (2021): To the Three Forefathers of Combinatorial Game Theory: The John Conway, Richard Guy, and Elwyn Berlekamp Memorial Volume, A9 (18 pages)

Image citation

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